ON THE

STABILITY OF THE MOTION

OF

SATURN'S RINGS.

AN ESSAY,

WHICH OBTAINED THE ADAMS PRIZE FOR THE YEAR 1856, IN THE UNIVERSITY OF CAMBRIDGE.

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"*E pur si muove.*'

Cambridge:

MACMILLAN AND CO.

AND 23 HENRIETTA STREET, COVENT GARDEN, LONDON.

1859.

ON THE MUTUAL PERTURBATIONS OF TWO RINGS.

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THERE are some questions in Astronomy, to which we are attracted rather on account of their peculiarity, as the possible illustration of some unknown principle, than from any direct advantage which their solution would afford to mankind. The theory of the Moon's inequalities, though in its first stages it presents theorems interesting to all students of mechanics, has been pursued into such intricacies of calculation as can be followed up only by those who make the improvement of the Lunar Tables the object of their lives. The value of the labours of these men is recognised by all who are aware of the importance of such tables in Practical Astronomy and Navigation. The methods by which the results are obtained are admitted to be sound, and we leave to professional astronomers the labour and the merit of developing them.

The questions which are suggested by the appearance of Saturn's Rings cannot, in the present state of Astronomy, call forth so great an amount of labour among mathematicians. I am not aware that any practical use has been made of Saturn's Rings, either in Astronomy or in Navigation. They are too distant, and too insignificant in mass, to produce any appreciable effect on the motion of other parts of the Solar system; and for this very reason it is difficult to determine those elements of their motion which we obtain so accurately in the case of bodies of greater mechanical importance.

But when we contemplate the Rings from a purely scientific point of view, they become the most remarkable bodies in the heavens, except, perhaps, those still less *useful* bodies—the spiral nebulæ. When we have actually seen that great arch swung over the equator of the planet without any visible connexion, we cannot bring our minds to rest. We cannot simply admit that such is the case, and describe it as one of the observed facts in nature, not admitting or requiring explanation. We must either explain its motion on the principles of mechanics, or admit that, in the Saturnian realms, there can be motion regulated by laws which we are unable to explain.

The arrangement of the rings is represented in the figure (1) on a scale of one inch to a hundred thousand miles. *S* is a section of Saturn through his equator, *A*, *B* and *C* are the three rings. *A* and *B* have been known for 200 years. They were mistaken by Galileo for protuberances on the planet itself, or perhaps satellites. Huyghens discovered that what he saw was a thin flat ring not touching the planet, and Ball discovered the division between *A* and *B*. Other divisions have been observed splitting these again into concentric rings, but these have not continued visible, the only well-established division being one in the middle of *A*. The third ring *C* was first detected by Mr Bond, at Cambridge, U. S. on November 15, 1850; Mr Dawes, not aware of Mr Bond's discovery, observed it on November 29th, and Mr Lassel a few days later. It gives little light compared with the other rings, and is seen where it crosses the planet as an obscure belt, but it is so transparent that the limb of the planet is visible through it, and this without distortion, showing that the rays of light have not passed through a transparent substance, but between the scattered particles of a discontinuous stream.

It is difficult to estimate the thickness of the system; according to the best estimates it is not more than 100 miles, the diameter of *A* being 176,418 miles; so that on the scale of our figure the thickness would be one thousandth of an inch.

Such is the scale on which this magnificent system of concentric rings is constructed; we have next to account for their continued existence, and to reconcile it with the known laws of motion and gravitation, so that by rejecting every hypothesis which leads to conclusions at variance with the facts, we may learn more of the nature of these distant bodies than the telescope can yet ascertain. We must account for the rings remaining suspended above the planet, concentric with Saturn and in his equatoreal plane; for the flattened figure of the section of each ring, for the transparency of the inner ring, and for the gradual approach of the inner edge of the ring to the body of Saturn as deduced from all the recorded observations by M. Otto Struvé (*Sur les dimensions des Anneaux de Saturne*—Recueil de Mémoires Astronomiques, Poulkowa. 15 Nov. 1851). For an account of the general appearance of the rings as seen from the planet, see Lardner on the Uranography of Saturn, *Mem. of the Astronomical Society*, 1853. See also the article "Saturn" in Nichol's *Cyclopædia of the Physical Sciences*.

Our curiosity with respect to these questions is rather stimulated than appeased by the investigations of Laplace. That great mathematician, though occupied with many questions which more imperiously demanded his attention, has devoted several chapters in various parts of his great work, to points connected with the Saturnian System.

He has investigated the law of attraction of a ring of small section on a point very near it, (*Méc. Cél.* Liv. III. Chap. VI.), and from this he deduces the equation from which the ratio of the breadth to the thickness of each ring is to be found,

$$e = \frac{R^3}{3a^3}\frac{\rho}{\rho'} = \frac{\lambda(\lambda - 1)}{(\lambda + 1)(3\lambda^2 + 1)},$$

where R is the radius of Saturn, and ρ his density; a the radius of the ring, and ρ' its density; and λ the ratio of the breadth of the ring to its thickness. The equation for determining λ when e is given has one negative root which must be rejected, and two roots which are positive while $e < 0{\cdot}0543$, and impossible when e has a greater value. At the critical value of e, $\lambda = 2{\cdot}594$ nearly.

The fact that λ is impossible when e is above this value, shows that the ring cannot hold together if the ratio of the density of the planet to that of the ring exceeds a certain value. This value is estimated by Laplace at $1{\cdot}3$, assuming $a = 2R$.

We may easily follow the physical interpretation of this result, if we observe that the forces which act on the ring may be reduced to—

(1) The attraction of Saturn, varying inversely as the square of the distance from his centre.

(2) The centrifugal force of the particles of the ring, acting outwards, and varying directly as the distance from Saturn's polar axis.

(3) The attraction of the ring itself, depending on its form and density, and directed, roughly speaking, towards the centre of its section.

The first of these forces must balance the second somewhere near the mean distance of the ring. Beyond this distance their resultant will be outwards, within this distance it will act inwards.

If the attraction of the ring itself is not sufficient to balance these residual forces, the outer and inner portions of the ring will tend to separate, and the ring will be split up; and it appears from Laplace's result that this will be the case if the density of the ring is less than $\frac{10}{13}$ of that of the planet.

This condition applies to all rings whether broad or narrow, of which the parts are separable, and of which the outer and inner parts revolve with the same angular velocity.

Laplace has also shown (Liv. v. Chap. III.), that on account of the oblateness of the figure of Saturn, the planes of the rings will follow that of Saturn's equator through every change of its position due to the disturbing action of other heavenly bodies.

Besides this, he proves most distinctly (Liv. III. Chap. VI.), that a solid uniform ring cannot possibly revolve about a central body in a permanent manner, for the slightest displacement of the centre of the ring from the centre of the planet would originate a motion which would never be checked, and would inevitably precipitate the ring upon the planet, not necessarily by breaking the ring, but by the inside of the ring falling on the equator of the planet.

He therefore infers that the rings are irregular solids, whose centres of gravity do not coincide with their centres of figure. We may draw the conclusion more formally as follows, "If the rings were solid and uniform, their motion would be unstable, and they would be destroyed. But they are not destroyed, and their motion is stable; therefore they are either not uniform or not solid."

I have not discovered* either in the works of Laplace or in those of more recent

* Since this was written, Prof. Challis has pointed out to me three important papers in Gould's *Astronomical Journal*:—Mr G. P. Bond *on the Rings of Saturn* (May 1851) and Prof. B. Pierce of Harvard University *on the Constitution of Saturn's Rings* (June 1851), and *on the Adams' Prize Problem* for 1856 (Sept. 1855). These American mathematicians have both considered the conditions of statical equilibrium of a transverse section of a ring, and have come to the conclusion that the rings, if they

mathematicians, any investigation of the motion of a ring either not uniform or not solid. So that in the present state of mechanical science, we do not know whether an irregular solid ring, or a fluid or disconnected ring can revolve permanently about a central body; and the Saturnian system still remains an unregarded witness in heaven to some necessary, but as yet unknown, development of the laws of the universe.

We know, since it has been demonstrated by Laplace, that a uniform solid ring cannot revolve permanently about a planet. We propose in this Essay to determine the amount and nature of the irregularity which would be required to make a permanent rotation possible. We shall find that the stability of the motion of the ring would be ensured by loading the ring at one point with a heavy satellite about $4\frac{1}{2}$ times the weight of the ring, but this load, besides being inconsistent with the observed appearance of the rings, must be far too artificially adjusted to agree with the natural arrangements observed elsewhere, for a very small error in excess or defect would render the ring again unstable.

We are therefore constrained to abandon the theory of a solid ring, and to consider the case of a ring, the parts of which are not rigidly connected, as in the case of a ring of independent satellites, or a fluid ring.

There is now no danger of the whole ring or any part of it being precipitated on the body of the planet. Every particle of the ring is now to be regarded as a satellite of Saturn, disturbed by the attraction of a ring of satellites at the same mean distance from the planet, each of which however is subject to slight displacements. The mutual action of the parts of the ring will be so small compared with the attraction of the planet, that no part of the ring can ever cease to move round Saturn as a satellite.

But the question now before us is altogether different from that relating to the solid ring. We have now to take account of variations in the form and arrangement of the parts of the ring, as well as its motion as a whole, and we have as yet no security that these variations may not accumulate till the ring entirely loses its original form, and collapses into one or more satellites, circulating round Saturn. In fact such a result is one of the leading doctrines of the "nebular theory" of the formation of planetary systems: and we are familiar with the actual breaking up of fluid rings under the action of "capillary" force, in the beautiful experiments of M. Plateau.

In this essay I have shewn that such a destructive tendency actually exists, but that by the revolution of the ring it is converted into the condition of dynamical stability. As the

move each as a whole, must be very narrow compared with the observed rings, so that in reality there must be a great number of them, each revolving with its own velocity. They have also entered on the question of the fluidity of the rings, and Prof. Pierce has made an investigation as to the permanence of the motion of an irregular solid ring and of a fluid ring. The paper in which these questions are treated at large has not (so far as I am aware) been published, and the references to it in Gould's Journal are intended to give rather a popular account of the results, than an accurate outline of the methods employed. In treating of the attractions of an irregular ring, he makes admirable use of the theory of potentials, but his published investigation of the motion of such a body contains some oversights which are due perhaps rather to the imperfections of popular language than to any thing in the mathematical theory. The only part of the theory of a fluid ring which he has yet given an account of, is that in which he considers the form of the ring at any instant as an ellipse; corresponding to the case where $n=\omega$, and $m=1$. As I had only a limited time for reading these papers, and as I could not ascertain the methods used in the original investigations, I am unable at present to state how far the results of this essay agree with or differ from those obtained by Professor Pierce.

scientific interest of Saturn's Rings depends at present mainly on this question of their stability, I have considered their motion rather as an illustration of general principles, than as a subject for elaborate calculation, and therefore I have confined myself to those parts of the subject which bear upon the question of the permanence of a given form of motion.

There is a very general and very important problem in Dynamics, the solution of which would contain all the results of this Essay and a great deal more. It is this—

"Having found a particular solution of the equations of motion of any material system, to determine whether a slight disturbance of the motion indicated by the solution would cause a small periodic variation, or a total derangement of the motion."

The question may be made to depend upon the conditions of a maximum or a minimum of a function of many variables, but the theory of the tests for distinguishing maxima from minima by the Calculus of Variations becomes so intricate when applied to functions of several variables, that I think it doubtful whether the physical or the abstract problem will be first solved.

PART I.

ON THE MOTION OF A RIGID BODY OF ANY FORM ABOUT A SPHERE.

We confine our attention for the present to the motion in the plane of reference, as the interest of our problem belongs to the character of this motion, and not to the librations, if any, from this plane.

Let S (Fig. 2) be the centre of gravity of the sphere, which we may call Saturn, and R that of the rigid body, which we may call the Ring. Join RS, and divide it in G so that

$$SG : GR :: R : S,$$

R and S being the masses of the Ring and of Saturn respectively.

Then G will be centre of gravity of the system, and its position will be unaffected by any mutual action between the parts of the system. Assume G as the point to which the motions of the system are to be referred. Draw GA in a direction fixed in space.

$$\text{Let } AGR = \theta, \text{ and } SR = r, \text{ then}$$

$$GR = \frac{S}{S+R}r, \text{ and } GS = \frac{R}{S+R}r,$$

so that the positions of S and R are now determined.

Let BRB' be a straight line through R, *fixed with respect to the substance of the ring*, and let $BRK = \phi$.

This determines the angular position of the ring, so that from the values of r, θ, and ϕ the configuration of the system may be deduced, as far as relates to the plane of reference.

We have next to determine the forces which act between the ring and the sphere, and this we shall do by means of the *potential function* due to the ring, which we shall call V.

The value of V for any point of space S, depends on its position relatively to the ring, and it is found from the equation

$$V = \Sigma\left(\frac{dm}{r'}\right),$$

where dm is an element of the mass of the ring, and r' is the distance of that element from the given point, and the summation is extended over every element of mass belonging to the ring. V will then depend entirely upon the position of the point S relatively to the ring, and may be expressed as a function of r, the distance of S from R, the centre of gravity of the ring, and ϕ, the angle which the line SR makes with the line RB, fixed in the ring.

A particle P, placed at S, will, by the theory of potentials, experience a moving force $P\dfrac{dV}{dr}$ in the direction which tends to increase r, and $P\dfrac{1}{r}\dfrac{dV}{d\phi}$ in a tangential direction, tending to increase ϕ.

Now we know that the attraction of a sphere is the same as that of a particle of equal mass placed at its centre. The forces acting between the sphere and the ring are therefore $S\dfrac{dV}{dr}$ tending to increase r, and a tangential force $S\dfrac{1}{r}\dfrac{dV}{d\phi}$, applied at S tending to increase ϕ. In estimating the effect of this latter force on the ring, we must resolve it into a tangential force $S\dfrac{1}{r}\dfrac{dV}{d\phi}$ acting at R, and a couple $S\dfrac{dV}{d\phi}$ tending to increase ϕ.

We are now able to form the equations of motion for the planet and the ring.

For the planet

$$S\frac{d}{dt}\left\{\left(\frac{Rr}{S+R}\right)^2\frac{d\theta}{dt}\right\} = -\frac{R}{S+R}S\frac{dV}{d\phi} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1),$$

$$S\frac{d^2}{dt^2}\left(\frac{Rr}{S+R}\right) - S\frac{Rr}{S+R}\left(\frac{d\theta}{dt}\right)^2 = S\frac{dV}{dr} \quad . \; . \; . \; . \; . \; . \; . \quad (2).$$

For the centre of gravity of the ring,

$$R\frac{d}{dt}\left\{\left(\frac{Sr}{S+R}\right)^2\frac{d\theta}{dt}\right\} = -\frac{S}{S+R}S\frac{dV}{d\phi} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3),$$

$$R\frac{d^2}{dt^2}\left(\frac{Sr}{S+R}\right) - R\frac{Sr}{S+R}\left(\frac{d\theta}{dt}\right)^2 = S\frac{dV}{dr} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (4).$$

For the rotation of the ring about its centre of gravity,

$$Rk^2\frac{d^2}{dt^2}(\theta+\phi) = S\frac{dV}{d\phi} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5),$$

where k is the radius of gyration of the ring about its centre of gravity.

Equations (3) and (4) are necessarily identical with (1) and (2), and show that the orbit of the centre of gravity of the ring must be similar to that of the Planet. Equations (1) and (3) are equations of areas, (2) and (4) are those of the radius vector.

Equations (3) (4) and (5) may be thus written,

$$R\left\{2r\frac{dr}{dt}\frac{d\theta}{dt} + r^2\frac{d^2\theta}{dt^2}\right\} + (R+S)\frac{dV}{d\phi} = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (6),$$

$$R\left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\} - (R+S)\frac{dV}{dr} \qquad = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (7),$$

$$Rk^2\left(\frac{d^2\theta}{dt^2} + \frac{d^2\phi}{dt^2}\right) - S\frac{dV}{d\phi} \qquad = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (8).$$

These are the necessary and sufficient data for determining the motion of the ring, the initial circumstances being given.

Prob. I. To find the conditions under which a uniform motion of the ring is possible.

By a uniform motion is here meant a motion of uniform rotation, during which the position of the centre of the Planet with respect to the ring does not change.

In this case r and ϕ are constant, and therefore V and its differential coefficients are given. Equation (7) becomes,

$$Rr\left(\frac{d\theta}{dt}\right)^2 + (R+S)\frac{dV}{dr} = 0,$$

which shows that the angular velocity is constant, and that

$$\left(\frac{d\theta}{dt}\right)^2 = -\frac{R+S}{Rr}\frac{dV}{dr} = \omega^2, \text{ say} \quad \ldots\ldots\ldots \quad (9).$$

Hence, $\frac{d^2\theta}{dt^2} = 0$, and therefore by equation (8),

$$\frac{dV}{d\phi} = 0 \quad \ldots\ldots\ldots\ldots \quad (10).$$

Equations (9) and (10) are the conditions under which the uniform motion is possible, and if they were exactly fulfilled, the uniform motion would go on for ever if not disturbed. But it does not follow that if these conditions were *nearly* fulfilled, or that if when accurately adjusted, the motion were *slightly* disturbed, the motion would go on for ever *nearly* uniform. The effect of the disturbance might be either to produce a periodic variation in the elements of the motion, the amplitude of the variation being small, or to produce a displacement which would increase indefinitely, and derange the system altogether. In the one case the motion would be *dynamically stable*, and in the other it would be *dynamically unstable*. The investigation of these displacements while still very small will form the next subject of inquiry.

Prob. II. To find the equations of the motion when slightly disturbed.

Let $r = r_0$, $\theta = \omega t$ and $\phi = \phi_0$ in the case of uniform motion, and let

$$r = r_0 + r_{,},$$
$$\theta = \omega t + \theta_{,},$$
$$\phi = \phi_0 + \phi_{,},$$

when the motion is slightly disturbed, where $r_{,}$ $\theta_{,}$ and $\phi_{,}$ are to be treated as small quantities of the first order, and their powers and products are to be neglected. We may expand $\frac{dV}{dr}$ and $\frac{dV}{d\phi}$ by Taylor's Theorem,

$$\frac{dV}{dr} = \frac{dV}{dr} + \frac{d^2V}{dr^2}r_{,} + \frac{d^2V}{dr\,d\phi}\phi_{,},$$

$$\frac{dV}{d\phi} = \frac{dV}{d\phi} + \frac{d^2V}{dr\,d\phi}r_{,} + \frac{d^2V}{d\phi^2}\phi_{,},$$

where the values of the differential coefficients on the right-hand side of the equations are those in which r_0 stands for r, and ϕ_0 for ϕ.

Calling
$$\frac{d^2V}{dr^2} = L, \quad \frac{d^2V}{dr\,d\phi} = M, \quad \frac{d^2V}{d\phi^2} = N,$$

and taking account of equations (9) and (10), we may write these equations,

$$\frac{dV}{dr} = -\frac{Rr_0}{R+S}\omega^2 + Lr_1 + M\phi_1,$$

$$\frac{dV}{d\phi} = Mr_1 + N\phi_1.$$

Substituting these values in equations (6), (7), (8), and retaining all small quantities of the first order while omitting their powers and products, we have the following system of linear equations in r_1 θ_1 and ϕ_1,

$$R\left(2r_0\omega\frac{dr_1}{dt} + r_0^2\frac{d^2\theta}{dt^2}\right) + (R+S)(Mr_1 + N\phi_1) = 0 \quad . \quad . \quad . \quad . \quad (11),$$

$$R\left(\frac{d^2r_1}{dt^2} - \omega^2 r_1 - 2r_0\omega\frac{d\theta_1}{dt}\right) - (R+S)(Lr_1 + M\phi_1) = 0 \quad . \quad . \quad . \quad . \quad (12),$$

$$Rk^2\left(\frac{d^2\theta_1}{dt^2} + \frac{dt^2}{dt^2}\right) - S(Mr_1 + N\phi_1) \qquad = 0 \quad . \quad . \quad . \quad . \quad (13).$$

Prob. III. To reduce the three simultaneous equations of motion to the form of a single linear equation.

Let us write n instead of the symbol $\frac{d}{dt}$, then arranging the equations in terms of r_1 θ_1 and ϕ_1, they may be written:

$$\left\{2Rr_0\omega n + (R+S)M\right\}r_1 + (Rr_0^2n^2)\theta_1 + (R+S)N\phi_1 = 0 \quad . \quad . \quad . \quad (14),$$

$$\left\{Rn^2 - R\omega^2 - (R+S)L\right\}r_1 - (2Rr_0\omega n)\theta_1 - (R+S)M\phi_1 = 0 \quad . \quad . \quad . \quad (15),$$

$$-(SM)r_1 + (Rk^2n^2)\theta_1 + (Rk^2n^2 - SN)\phi_1 = 0 \quad . \quad . \quad . \quad (16).$$

Here we have three equations to determine three quantities r_1 θ_1 ϕ_1; but it is evident that only a relation can be determined between them, and that in the process for finding their absolute values, the three quantities will vanish together, and leave the following relation among the coefficients,

$$\left.\begin{aligned}
&-\left\{2Rr_0\omega n + (R+S)M\right\}\left\{2Rr_0\omega n\right\}\left\{Rk^2n^2 - SN\right\} \\
&+\left\{Rn^2 - R\omega^2 - (R+S)L\right\}\left\{Rk^2n^2\right\}\left\{(R+S)N\right\} \\
&+(SM)(Rr_0^2n^2)(R+S)M - (SM)(2Rr_0\omega n)(R+S)N \\
&+\left\{2Rr_0\omega n + (R+S)M\right\}\left\{Rk^2n^2\right\}\left\{(R+S)M\right\} \\
&-\left\{Rn^2 - R\omega^2 - (R+S)\right\}\left\{Rr_0^2n^2\right\}\left\{Rk^2n^2 - SN\right\}
\end{aligned}\right\} = 0 \quad . \quad . \quad . \quad (17).$$

By multiplying up, and arranging by powers of n and dividing by Rn^2, this equation becomes

$$An^4 + Bn^2 + C = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

where

$$\left.\begin{aligned} A &= R^2 r_0^{\ 2} k^2, \\ B &= 3R^2 r_0^{\ 2} k^2 \omega^2 - R\,(R+S)\,L r_0^{\ 2} k^2 - R\,\{(R+S)\,k^2 + Sr^2\}\,N \\ C &= R\,\{(R+S)\,k^2 - 3Sr_0^{\ 2}\}\,\omega^2 + (R+S)\,\{(R+S)\,k^2 + Sr_0^{\ 2}\}\,(LN - M^2) \end{aligned}\right\} \quad . \quad . \quad (19).$$

Here we have a biquadratic equation in n which may be treated as a quadratic in n^2, it being remembered that n stands for the operation $\frac{d}{dt}$.

Prob. IV. To determine whether the motion of the ring is stable or unstable, by means of the relations of the coefficients A, B, C.

The equations to determine the forms of r_1 θ_1 ϕ_1 are all of the form

$$A\frac{d^4u}{dt^4} + B\frac{d^2u}{dt^2} + Cu = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

and if n be one of the four roots of equation (18), then

$$u = Ce^{nt}$$

will be one of the four terms of the solution, and the values of r_1 θ_1 and ϕ_1 will differ only in the values of the coefficient C.

Let us inquire into the nature of the solution in different cases.

(1) If n be positive, this term would indicate a displacement which must increase indefinitely, so as to destroy the arrangement of the system.

(2) If n be negative, the disturbance which it belongs to would gradually die away.

(3) If n be a pure impossible quantity, of the form $\pm a\sqrt{-1}$, then there will be a term in the solution of the form $C\cos(at + \alpha)$, and this would indicate a periodic variation, whose amplitude is C. and period $\frac{2\pi}{a}$.

(4) If n be of the form $b \pm \sqrt{-1}\,a$, the first term being positive and the second impossible, there will be a term in the solution of the form

$$C\epsilon^{bt}\cos(at + \alpha),$$

which indicates a periodic disturbance, whose amplitude continually increases till it disarranges the system.

(5) If n be of the form $-b \pm \sqrt{-1}\,a$, a negative quantity and an impossible one, the corresponding term of the solution is

$$C\epsilon^{-bt}\cos(at + \alpha),$$

which indicates a periodic disturbance whose amplitude is constantly diminishing.

It is manifest that the first and fourth cases are inconsistent with the permanent motion of the system. Now since equation (18) contains only even powers of n, it must have pairs of equal and opposite roots, so that every root coming under the second or fifth cases, implies the existence of another root belonging to the first or fourth. If such a root exists, some disturbance may occur to produce the kind of derangement corresponding to it, so that the system is not safe unless roots of the first and fourth kinds are altogether excluded. This cannot be done without excluding those of the second and fifth kinds, so that, to insure stability, all the four roots must be of the third kind, that is, pure impossible quantities.

That this may be the case, both values of n^2 must be real and negative, and the conditions of this are—

1st. That A, B, and C should be of the same sign,

2ndly. That $B^2 > 4AC$.

When these conditions are fulfilled, the disturbances will be periodic and consistent with stability. When they are not both fulfilled, a small disturbance may produce total derangement of the system.

Prob. V. To find the centre of gravity, the radius of gyration, and the variations of the potential near the centre of a circular ring of small but variable section.

Let a be the radius of the ring, and let θ be the angle subtended at the centre between the radius through the centre of gravity and the line through a given point in the ring. Then if μ be the mass of unit of length of the ring near the given point, μ will be a periodic function of θ, and may therefore be expanded by Fourier's theorem in the series,

$$\mu = \frac{R}{2\pi a}\left\{1 + 2f\cos\theta + \frac{2}{3}g\cos 2\theta + \frac{2}{3}h\sin 2\theta + 2i\cos(3\theta + a) + \&c.\right\} \quad . \quad . \qquad (21),$$

where f, g, h, &c. are arbitrary coefficients, and R is the mass of the ring.

(1) The moment of the ring about the diameter perpendicular to the prime radius is

$$Rr_0 = \int_0^{2\pi} \mu a^2 \cos\theta d\theta = Raf,$$

therefore the distance of the centre of gravity from the centre of the ring,

$$r_0 = af.$$

(2) The radius of gyration of the ring about its centre in its own plane is evidently the radius of the ring $= a$, but if k be that about the centre of gravity, we have

$$k^2 + r_0^2 = a^2;$$

$$\therefore \; k^2 = a^2(1 - f^2).$$

(3) The potential at any point is found by dividing the mass of each element by its distance from the given point, and integrating over the whole mass.

Let the given point be near the centre of the ring, and let its position be defined by the co-ordinates r' and ψ, of which r' is small compared with a.

The distance (ρ) between this point and a point in the ring is

$$\frac{1}{\rho} = \frac{1}{a}\left\{1 + \frac{r'}{a}\cos(\psi - \theta) + \frac{1}{4}\left(\frac{r'}{a}\right)^2 + \frac{3}{4}\left(\frac{r'}{a}\right)^2 \cos 2(\psi - \theta) + \&c.\right\}.$$

The other terms contain powers of $\frac{r}{a}$ higher than the second.

We have now to determine the value of the integral,

$$V = \int_0^{2\pi} \frac{\mu}{\rho}\, a d\theta\,;$$

and in multiplying the terms of (μ) by those of $\left(\frac{1}{\rho}\right)$, we need retain only those which contain constant quantities, for all those which contain sines or cosines of multiples of $(\psi - \theta)$ will vanish when integrated between the limits. In this way we find

$$V = \frac{R}{a}\left\{1 + f\frac{r'}{a}\cos\psi + \frac{1}{4}\frac{r'^2}{a^2}(1 + g\cos 2\psi + h\sin 2\psi)\right\} \quad . \quad . \quad . \quad (22).$$

The other terms containing higher powers of $\frac{r}{a}$.

In order to express V in terms of r_1 and ϕ_1, as we have assumed in the former investigation, we must put

$$r\cos\psi = -r_1 + \frac{1}{2}r_0\phi_1^2,$$

$$r\sin\psi = -r_0\phi_1^2,$$

$$V = \frac{R}{a}\left\{1 - f\frac{r_1}{a} + \frac{1}{4}\frac{r_1^2}{a^2}(1+g) + \frac{1}{2}\frac{r_1}{a}fr_1\phi_1 + \frac{1}{4}f^2\phi_1^2(3-g)\right\} \quad . \quad . \quad . \quad (23).$$

From which we find $\left(\frac{V}{dr}\right)_0 = -\frac{R}{a^2}f,$

$$\left.\begin{aligned}\left(\frac{d^2V}{dr^2}\right)_0 &= L = \frac{R}{2a^3}(1+g)\\ \left(\frac{d^2V}{drd\phi}\right)_0 &= M = \frac{R}{2a^2}\,fh\\ \left(\frac{d^2V}{d\phi^2}\right)_0 &= N = \frac{R}{2a}\,f^2(3-g)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

These results may be confirmed by the following considerations applicable to any circular ring, and not involving any expansion or integration. Let af be the distance of the centre of gravity from the centre of the ring, and let the ring revolve about its centre with velocity ω. Then the force necessary to keep the ring in that orbit will be $-Raf\omega^2$.

But let S be a mass fixed at the centre of the ring, then if

$$\omega^2 = \frac{S}{a^3},$$

every portion of the ring will be separately retained in its orbit by the attraction of S, so that the whole ring will be retained in its orbit. The resultant attraction must therefore pass through the centre of gravity, and be

$$-Raf\omega^2 = -RS\frac{f}{a^2};$$

$$\therefore \quad \frac{dV}{dr} = -R\frac{f}{a^2}.$$

The equation $$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} + 4\pi\rho = 0$$

is true for any system of matter attracting according to the law of gravitation. If we bear in mind that the expression is identical in form with that which measures the total efflux of fluid from a differential element of volume, when $\frac{dV}{dx}\,\frac{dV}{dy}\,\frac{dV}{dz}$ are the rates at which the fluid passes through its sides, we may easily form the equation for any other case. Now let the position of a point in space be determined by the co-ordinates r, ϕ and z, where z is measured perpendicularly to the plane of the angle ϕ. Then by choosing the directions of the axes xyz, so as to coincide with those of the radius vector r, the perpendicular to it in the plane of ϕ, and the normal, we shall have

$$dx = dr \qquad dz = rd\phi \qquad dy = dz$$

$$\frac{dV}{dx} = \frac{dV}{dr} \qquad \frac{dV}{dy} = \frac{1}{r}\,\frac{dV}{d\phi} \qquad \frac{dV}{dz} = \frac{dV}{dz}.$$

The quantities of fluid passing through an element of area in each direction are

$$\frac{dV}{dr}rd\phi dz, \quad \frac{dV}{d\phi}\,\frac{1}{r}drdz, \quad \frac{dV}{dz}rd\phi dr,$$

so that the expression for the whole efflux is

$$\frac{1}{r}\,\frac{dV}{dr} + \frac{d^2V}{dr^2} + \frac{1}{r^2}\,\frac{d^2V}{d\phi^2} + \frac{d^2V}{dz^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (25),$$

which is necessarily equivalent to the former expression.

Now at the centre of the ring $\frac{d^2V}{dz^2}$ may be found by considering the attraction on a point just above the centre at a distance z,

$$\frac{dV}{dz} = -R\frac{z}{(a^2+z^2)^{\frac{3}{2}}}$$

$$\frac{d^2V}{dz^2} = -\frac{R}{a^3} \text{ when } z = 0.$$

Also we know $$\frac{1}{r}\frac{dV}{dr} = -\frac{R}{a^3} \text{ and } r = af,$$

so that in any circular ring $$\frac{d^2V}{dr^2} + \frac{1}{a^2f^2}\frac{d^2V}{d\phi^2} = 2\frac{R}{a^3} \qquad (26),$$

an equation satisfied by the former values of L and N.

By referring to the original expression for the variable section of the ring, it appears that the effect of the coefficient f is to make the ring thicker on one side and thinner on the other in a uniformly graduated manner. The effect of g is to thicken the ring at two opposite sides, and diminish its section in the parts between. The coefficient h indicates an inequality of the same kind, only not symmetrically disposed about the diameter through the centre of gravity.

Other terms indicating inequalities recurring three or more times in the circumference of the ring, have no effect on the values of L, M and N. There is one remarkable case, however, in which the irregularity consists of a single heavy particle placed at a point on the circumference of the ring.

Let P be the mass of the particle, and Q that of the uniform ring on which it is fixed, then $R = P + Q$,

$$f = \frac{P}{R}$$

$$L = 2\frac{P}{a^3} + \frac{Q}{2a^3} = \frac{P+Q}{2a^2}\left(1 + 3\frac{P}{R}\right) = \frac{R}{2a^2}(1+g);$$

$$\therefore \quad g = \frac{3P}{R} = 3f \qquad (27).$$

Prob. VI. To determine the conditions of stability of the motion in terms of the coefficients f, g, h, which indicate the distribution of mass in the ring.

The quantities which enter into the differential equation of motion (18) are R, S, k^2, r_0, ω^2, L, M, N. We must observe that S is very large compared with R, and therefore we neglect R in those terms in which it is added to S, and we put

$$S = a^3\omega^2$$

$$k^2 = a^2(1 - f^2)$$

$$r_0 = af$$

$$L = \frac{R}{2a^3}(1+g)$$

$$M = \frac{R}{2a^2}fh$$

$$N = \frac{R}{2a}f^2(3-g).$$

Substituting these values in equation (18) and dividing by $R^2a^4f^2$, we obtain

$$(1-f^2)\,n^4 + \left(1-\frac{5}{2}f^2+\frac{1}{2}f^2g\right)n^2\omega^2 + \left(\frac{9}{4}-6f^2-\frac{1}{4}g^2-\frac{1}{4}h^2+2f^2g\right)\omega^4 = 0. \quad . \quad . \quad (28)$$

The condition of stability is that this equation shall give both values of n^2 negative. and this renders it necessary that all the coefficients should have the same sign, and that the square of the second should exceed four times the product of the first and third.

(1) Now if we suppose the ring to be uniform, f, g and h disappear, and the equation becomes

$$n^4 + n^2\omega^2 + \frac{9}{4} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29),$$

which gives impossible values to n^2 and indicates the instability of a uniform ring.

(2) If we make g and $h=0$, we have the case of a ring thicker at one side than the other, and varying in section according to the simple law of sines. We must remember, however, that f must be less than $\frac{1}{2}$, in order that the section of the ring at the thinnest part may be real. The equation becomes

$$(1-f^2)\,n^4 + \left(1-\frac{5}{2}f^2\right)n^2\omega^2 + \left(\frac{9}{4}-6f^2\right)\omega^4 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (30).$$

The condition that the third term should be positive gives

$$f^2 < \cdot 375.$$

The condition that n^2 should be real gives

$$71\,f^4 - 112\,f^2 + 32 \text{ negative},$$

which requires f^2 to be between ·37445 and 1·2.

The condition of stability is therefore that f^2 should lie between

·37445 and ·375,

but the construction of the ring on this principle requires that f^2 should be less than ·25, so that it is impossible to reconcile this form of the ring with the conditions of stability.

(3) Let us next take the case of a uniform ring, loaded with a heavy particle at a point of its circumference. We have then $g=3f$, $h=0$, and the equation becomes

$$(1-f^2)\,n^4 + \left(1-\frac{5}{2}f^2+\frac{3}{2}f^3\right)n^2\omega^2 + \left(\frac{9}{4}-\frac{33}{4}f^2+6f^3\right)\omega^4 = 0 \quad . \quad . \quad . \quad (31).$$

Dividing each term by $1-f$, we get

$$(1+f)\,n^4 + \left(1+f-\frac{3}{2}f^2\right)n^2\omega^2 + \frac{3}{4}\left\{3\,(1+f) - 8f^2\right\}\omega^4 = 0 \quad . \quad . \quad . \quad (32).$$

The first condition gives f less than ·8279.

The second condition gives f greater than ·815865.

Let us assume as a particular case between these limits $f = \cdot 82$, which makes the ratio of the mass of the particle to that of the ring as 82 to 18, then the equation becomes

$$1{\cdot}82\, n^4 + {\cdot}8114\, n^2\omega^2 + {\cdot}9696\, \omega^4 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33),$$

which gives $$\sqrt{-1}\, n = \pm {\cdot}5916\, \omega \text{ or } \pm {\cdot}3076.$$

These values of n indicate variations of r_1 θ_1 and ϕ_1, which are compounded of two simple periodic inequalities, the period of the one being 1·69 revolutions, and that of the other 3·251 revolutions of the ring. The relations between the phases and amplitudes of these inequalities must be deduced from equations (14), (15), (16), in order that the character of the motion may be completely determined.

Equations (14), (15), (16) may be written as follows:

$$(4n\omega + h\omega^2)\frac{r_1}{a} + 2fn^2\theta_1 + f(3-g)\,\omega^2\phi_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34),$$

$$\left\{n^2 - \frac{1}{2}\omega^2(3+g)\right\}\frac{r_1}{a} - 2f\omega n\theta_1 - \frac{1}{2}fh\omega^2\phi_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35),$$

$$-fh\omega^2\frac{r_1}{a} + 2(1-f^2)\,n^2\theta_1 + \left\{2(1-f^2)\,n^2 - f^2(3-g)\,\omega^2\right\}\phi_1 = 0 \quad . \quad (36).$$

By eliminating one of the variables between any two of these equations, we may determine the relation between the two remaining variables. Assuming one of these to be a periodic function of t of the form $A\cos\nu t$, and remembering that n stands for the operation $\frac{d}{dt}$, we may find the form of the other.

Thus, eliminating θ_1 between the first and second equations

$$\left\{n^3 + n\omega^2(5-g) + h\omega^3\right\}\frac{r}{a} + f\omega^3\left\{(3-g)\,\omega - \frac{1}{2}hn\right\}\phi_1 = 0. \quad . \quad . \quad . \quad (37).$$

Assuming $$\frac{r_1}{a} = A\sin\nu t, \text{ and } \phi_1 = Q\cos(\nu t - \beta)$$

$$\left\{-\nu^3 + \nu\omega^2(5-g)\right\}A\cos\nu t + h\omega^3 A\sin\nu t + f\omega^3(3-g)\,Q\cos(\nu t-\beta) + \frac{1}{2}fh\omega^2\nu Q\sin(\nu t-\beta).$$

Equating νt to 0, and to $\frac{\pi}{2}$, we get the equations

$$\left\{\nu^3 - \nu\omega^2(5-g)\right\}A = f\omega^2 Q\left\{(3-g)\,\omega\cos\beta - \frac{1}{2}h\nu\sin\beta\right\},$$

$$-h\omega^3 A = f\omega^2 Q\left\{(3-g)\,\omega\sin\beta + \frac{1}{2}h\nu\cos\beta\right\},$$

from which to determine Q and β.

In all cases in which the mass is disposed symmetrically about the diameter through the centre of gravity, $h = 0$ and the equations may be greatly simplified.

Let $\theta_1 = P\cos(\nu t + \alpha)$, then the second equation becomes

$$\left\{\nu^2 + \frac{1}{2}\omega^2(3+g)\right\} A \sin t = 2Pf\omega\nu \sin(\nu t + \alpha),$$

whence
$$\alpha = 0 \quad P = \frac{\nu^2 + \frac{1}{2}\omega^2(3+g)}{2f\omega\nu} A \quad \ldots\ldots \quad (38).$$

The first equation becomes

$$4A\omega\nu\cos\nu t - 2Pf\nu^2\cos\nu t + Qf(3-g)\omega^2\cos(\nu t - \beta) = 0,$$

whence
$$\beta = 0 \quad Q = \frac{\nu^3 - \frac{1}{2}\omega^2\nu(5-g)}{f(3-g)\omega^3} \quad \ldots\ldots \quad (39).$$

In the numerical example in which a heavy particle was fixed to the circumference of the ring, we have, when $f = \cdot 82$,

$$\frac{\nu}{\omega} = \begin{cases}\cdot 5916 \\ \cdot 3076\end{cases} \qquad \frac{P}{A} = \begin{cases}3\cdot 21 \\ 5\cdot 72\end{cases} \qquad \frac{Q}{A} = \begin{cases}-1\cdot 229 \\ -\cdot 797\end{cases},$$

so that if we put $\omega t = \theta_0$ = the mean anomaly,

$$\frac{r_1}{a} = A\sin(\cdot 5916\,\theta_0 - \alpha) + B\sin(\cdot 3076\,\theta_0 - \beta) \quad \ldots\ldots \quad (40),$$

$$\theta_1 = 3\cdot 21\,A\cos(\cdot 5916\,\theta_0 - \alpha) + 5\cdot 72\,B\cos(\cdot 3076\,\theta_0 - \beta) \quad \ldots\ldots \quad (41),$$

$$\phi_1 = -1\cdot 229\,A\cos(\cdot 5916\,\theta_0 - \alpha) - \cdot 797\,B\cos(\cdot 3076\,\theta_0 - \beta) \quad \ldots\ldots \quad (42).$$

These three equations serve to determine $r_1\ \theta_1$ and ϕ_1 when the original motion is given. They contain four arbitrary constants A, B, α, β. Now since the original values $r_1\ \theta_1\ \phi_1$, and also their first differential coefficients with respect to t are arbitrary, it would appear that six arbitrary constants ought to enter into the equation. The reason why they do not is, that we assume r_0 and θ_0 as the *mean values* of r and θ in the *actual motion*. These quantities therefore depend on the original circumstances, and the two additional arbitrary constants enter into the values of r_0 and θ_0. In the analytical treatment of the problem the differential equation in n was originally of the sixth degree with a solution

$$n^2 = 0,$$

which implies the possibility of terms in the solution of the form

$$Ct + D.$$

The existence of such terms depends on the previous equations, and we find that a term of this form may enter into the value of θ, and that r_1 may contain a constant term, but that in both cases these additions will be absorbed into the values of θ_0 and r_0.

PART II.

ON THE MOTION OF A RING, THE PARTS OF WHICH ARE NOT RIGIDLY CONNECTED.

1. In the case of the Ring of invariable form, we took advantage of the principle that the mutual actions of the parts of any system form at all times a system of forces in equilibrium, and we took no account of the attraction between one part of the ring and any other part, since no motion could result from this kind of action. But when we regard the different parts of the ring as capable of independent motion, we must take account of the attraction on each portion of the ring as affected by the irregularities of the other parts, and therefore we must begin by investigating the statical part of the problem in order to determine the forces that act on any portion of the ring, as depending on the instantaneous condition of the rest of the ring.

In order to bring the problem within the reach of our mathematical methods, we limit it to the case in which the ring is *nearly* circular and uniform, and has a transverse section very small compared with the radius of the ring. By analysing the difficulties of the theory of a linear ring, we shall be better able to appreciate those which occur in the theory of the actual rings.

The ring which we consider is therefore small in section, and very nearly circular and uniform, and revolving with nearly uniform velocity. The variations from circular form, uniform section, and uniform velocity must be expressed by a proper notation.

2. To express the position of an element of a variable ring at a given time in terms of the original position of the element in the ring.

Let S (fig. 3) be the central body, and SA a direction fixed in space.

Let SB be a radius, revolving with the mean angular velocity (ω) of the ring, so that $ASB = \omega t$.

Let π be an element of the ring in its actual position, and let P be the position it would have had if it had moved uniformly with the mean velocity ω and had not been displaced, then BSP is a constant angle $= s$, and the value of s enables us to identify any element of the ring.

The element may be removed from its mean position P in three different ways.

(1) By change of distance from S by a quantity $p\pi = \rho$.

(2) By change of angular position through a space $Pp = \sigma$.

(3) By displacement perpendicular to the plane of the paper by a quantity ζ.

ρ, σ and ζ are all functions of s and t. If we could calculate the attractions on any element as depending on the form of these functions, we might determine the motion of the ring for any given original disturbance. We cannot, however, make any calculations of this kind without knowing the form of the functions, and therefore we must adopt the following method of separating the original disturbance into others of simpler form, first given in Fourier's *Traité de Chaleur*.

3. Let U be a function of s, it is required to express U in a series of sines and cosines of multiples of s between the values $s = 0$ and $s = 2\pi$.

Assume
$$U = A_1 \cos s + A_2 \cos 2s + \text{\&c.} + A_m \cos ms + A_n \cos ns$$
$$+ B_1 \sin s + B_2 \cos 2s + \text{\&c.} + B_m \sin ms + B_n \sin ns.$$

Multiply by $\cos msds$ and integrate, then all terms of the form

$$\int \cos ms \cos nsds \quad \text{and} \quad \int \cos ms \sin nsds$$

will vanish, if we integrate from $s = 0$ to $s = 2\pi$, and there remains

$$\int_0^{2\pi} U \cos msds = \pi A_m, \qquad \int U \sin msds = \pi B_m.$$

If we can determine the values of these integrals in the given case, we can find the proper coefficients A_m, B_m, &c., and the series will then represent the values of U from $s = 0$ to $s = 2\pi$, whether those values be continuous or discontinuous, and when none of those values are infinite the series will be convergent.

In this way we may separate the most complex disturbances of a ring into parts whose form is that of a circular function of s or its multiples. Each of these partial disturbances may be investigated separately, and its effect on the attractions of the ring ascertained either accurately or approximately.

4. To find the magnitude and direction of the attraction between two elements of a disturbed ring.

Let P and Q (fig. 4) be the two elements, and let their original positions be denoted by s_1 and s_2, the values of the arcs BP, BQ before displacement. The displacement consists in the angle BSP being increased by σ_1 and BSQ by σ_2, while the distance of P from the centre is increased by ρ_1 and that of Q by ρ_2. We have to determine the effect of these displacements on the distance PQ and the angle SPQ.

Let the radius of the ring be unity, and $S_2 - S_1 = 2\theta$, then the original value of PQ will be $2 \sin \theta$, and the increase due to displacement

$$= (\rho_2 + \rho_1) \sin \theta + (\sigma_2 - \sigma_1) \cos \theta.$$

We may write the complete value of PQ thus,

$$PQ = 2 \sin \theta \left\{1 + \frac{1}{2}(\rho_2 + \rho_1) + \frac{1}{2}(\sigma_2 - \sigma_1) \cot \theta\right\} \quad . \quad . \quad . \quad . \quad (1).$$

The original value of the angle SPQ was $\frac{\pi}{2} - \theta$, and the increase due to displacement is

$$\frac{1}{2}(\rho_2 - \rho_1)\cot\theta - \frac{1}{2}(\sigma_2 - \sigma_1),$$

so that we may write the values of sin SPQ and cos SPQ,

$$\sin SPQ = \cos\theta\left\{1 + \frac{1}{2}(\rho_2 - \rho_1) - \frac{1}{2}(\sigma_2 - \sigma_1)\tan\theta\right\} \quad . \quad . \quad . \quad . \quad (2),$$

$$\cos SPQ = \sin\theta\left\{1 - \frac{1}{2}(\rho_2 - \rho_1)\cot^2\theta + \frac{1}{2}(\sigma_2 - \sigma_1)\cot\theta\right\} \quad . \quad . \quad (3).$$

If we assume the masses of P and Q each equal to $\frac{1}{\mu}R$, where R is the mass of the ring, and μ the number of satellites of which it is composed, the accelerating effect of the radial force on P is

$$\frac{1}{\mu}R\frac{\cos SPQ}{PQ^2} = \frac{1}{\mu}\frac{R}{4\sin\theta}\left\{1 - (\rho_2 + \rho_1) - \frac{1}{2}(\rho_2 - \rho_1)\cot^2\theta - \frac{1}{2}(\sigma_2 - \sigma_1)\cot\theta\right\} \quad (4),$$

and the tangential force

$$\frac{1}{\mu}R\frac{\sin SPQ}{PQ^2} = \frac{1}{\mu}\frac{R\cos\theta}{4\sin^2\theta}\left\{1 - \frac{1}{2}\rho_2 - \frac{3}{2}\rho_1 - (\sigma_2 - \sigma_1)\left(\cot\theta + \frac{1}{2}\tan\theta\right)\right\} \quad . \quad . \quad (5).$$

The normal force is $\frac{1}{\mu}R\frac{\zeta_2 - \zeta_1}{8\sin^3\theta}$.

5. Let us substitute for ρ, σ and ζ their values expressed in a series of sines and cosines of multiples of s, the terms involving ms being

$$\begin{aligned}
\rho_1 &= A\cos(ms + \alpha), & \rho_2 &= A\cos(ms + \alpha + 2\theta),\\
\sigma_1 &= B\sin(ms + \beta), & \sigma_2 &= B\sin(ms + \beta + 2\theta),\\
\zeta_1 &= C\cos(ms + \gamma), & \zeta_2 &= C\cos(ms + \gamma + 2\theta).
\end{aligned}$$

The radial force now becomes

$$\frac{1}{\mu}\frac{R}{4\sin\theta}\left\{\begin{aligned}&1 - A\cos(ms + \alpha)(1 + \cos 2m\theta) + A\sin(ms + \alpha)\sin 2m\theta\\ &+ \tfrac{1}{2}A\cos(ms + \alpha)(1 - \cos 2m\theta)\cot^2\theta - \tfrac{1}{2}A\sin(ms + \alpha)\sin 2m\theta\cot^2\theta\\ &+ \tfrac{1}{2}B\sin(ms + \beta)(1 - \cos 2m\theta)\cot\theta - \tfrac{1}{2}B\cos(ms + \beta)\sin 2m\theta\cot\theta\end{aligned}\right\} \quad (6).$$

The radial component of the attraction of a corresponding particle on the other side of P may be found by changing the sign of θ. Adding the two together, we have for the effect of the pair

$$\frac{1}{\mu}\frac{R}{2\sin\theta}\left\{1 - A\cos(ms + \alpha)(2\cos^2 m\theta - \sin^2 m\theta\cot^2\theta)\right.$$

$$\left. - B\cos(ms + \beta)\frac{1}{2}\sin 2m\theta\cot\theta\right\} \quad . \quad . \quad . \quad . \quad (7).$$

Let us put

$$\left.\begin{aligned} L &= \Sigma\left(\frac{1}{2}\frac{\sin^2 m\theta\cos^2\theta}{\sin^3\theta} - \frac{\cos^2 m\theta}{\sin\theta}\right) \\ M &= \Sigma\left(\frac{\sin 2m\theta\cos\theta}{4\sin^2\theta}\right) \\ N &= \Sigma\left(\frac{\sin^2 m\theta\cos^2\theta}{\sin^3\theta} + \frac{1}{2}\frac{\sin^2 m\theta}{\sin\theta}\right) \\ J &= \Sigma\left(\frac{\sin^2 m\theta}{2\sin^3\theta}\right) \\ K &= \Sigma\left(\frac{1}{2\sin\theta}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (8)^*,$$

where the summation extends to all the satellites on the same side of P, that is, every value of θ of the form $\frac{x}{\mu}\pi$, where x is a whole number less than $\frac{\mu}{2}$.

The radial force may now be written

$$P = \frac{1}{\mu}R\left\{K + LA\cos(ms+\alpha) - MB\cos(ms+\beta)\right\} \quad \ldots\ldots \quad (9).$$

The tangential force may be calculated in the same way, it is

$$T = \frac{1}{\mu}R\left\{MA\sin(ms+\alpha) + NB\sin(ms+\beta)\right\} \quad \ldots\ldots \quad (10).$$

The normal force is

$$Z = -\frac{1}{\mu}RJC\cos(ms+\gamma) \quad \ldots\ldots\ldots \quad (11).$$

6. We have found the expressions for the forces which act upon each member of a system of equal satellites which originally formed a uniform ring, but are now affected with displacements depending on circular functions. If these displacements can be propagated round the ring in the form of waves with the velocity $\frac{m}{n}$, the quantities α, β and γ will depend on t, and the complete expressions will be

$$\left.\begin{aligned} \rho &= A\cos(ms+nt+\alpha) \\ \sigma &= B\sin(ms+nt+\beta) \\ \zeta &= C\cos(ms+n't+\gamma) \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (12).$$

* The following values of several quantities which enter into these investigations are calculated for a ring of 36 satellites.

$K = 24{\cdot}5$

	$\Sigma\frac{\sin^2 m\theta\cos^2\theta}{\sin^3\theta}$,	$\Sigma\frac{\cos^2 m\theta}{\sin\theta}$	L	M	N
$m=0$	0	43	−43	0	0
$m=1$	32	32	−16	16	37
$m=2$	107	28	26	25	115
$m=3$	212	25	81	28	221
$m=4$	401	24	177	32	411
$m=9$	975	20	468	30	986
$m=18$	1569	18	767	0	1582

When μ is very great,

$$\left.\frac{\pi}{\mu}\right]^3 L = .5259 \text{ when } m = \frac{\mu}{2},$$

$$= .4342 \qquad m = \frac{\mu}{3},$$

$$= .3287 \qquad m = \frac{\mu}{4}.$$

Let us find in what cases expressions such as these will be true, and what will be the result when they are not true.

Let the position of a satellite at any time be determined by the values of r, ϕ and ζ, where r is the radius vector reduced to the plane of reference, ϕ the angle of position measured on that plane, and ζ the distance from it. The equations of motion will be

$$\left.\begin{aligned} r\left(\frac{d\phi}{dt}\right)^2 - \frac{d^2r}{dt^2} &= S\frac{1}{r^2} + P \\ 2\frac{dr}{dt}\frac{d\phi}{dt} + r\frac{d^2\phi}{dt^2} &= T \\ \frac{d^2\zeta}{dt^2} &= -S\frac{\zeta}{r^3} + Z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (13).$$

If we substitute the value of ζ in the third equation and remember that r is nearly $= 1$, we find

$$n'^2 = S + \frac{1}{\mu}RJ \quad \ldots\ldots\ldots\ldots \quad (14).$$

As this expression is necessarily positive, the value of n' is always real, and the disturbances normal to the plane of the ring can always be propagated as waves, and therefore can never be the cause of instability. We therefore confine our attention to the motion in the plane of the ring as deduced from the two former equations.

Putting $r = 1 + \rho$ and $\phi = \omega t + s + \sigma$, and omitting powers and products of ρ, σ and their differential coefficients,

$$\left.\begin{aligned} \omega^2 + \omega^2\rho + 2\omega\frac{d\sigma}{dt} - \frac{d^2\rho}{dt^2} &= S - 2S\rho + P \\ 2\omega\frac{d\rho}{dt} + \frac{d^2\sigma}{dt^2} &= T \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (15).$$

Substituting the values of ρ and σ as given above, these equations become

$$\omega^2 - S - \frac{1}{\mu}RK + (\omega^2 + 2S - \frac{1}{\mu}RL + n^2)A\cos(ms + nt + a)$$
$$+ (2\omega n + \frac{1}{\mu}RM)B\cos(ms + nt + \beta) = 0 \quad \ldots \quad (16),$$

$$(2\omega n + \frac{1}{\mu}RM)A\sin(ms + nt + a) + (n^2 + \frac{1}{\mu}RN)B\sin(ms + nt + \beta) = 0 \quad . \quad (17).$$

Putting for $(ms + nt)$ any two different values, we find from the second equation

$$a = \beta \quad \ldots\ldots\ldots\ldots \quad (18),$$

and

$$(2\omega n + \frac{1}{\mu}RM)A + (n^2 + \frac{1}{\mu}RN)B = 0 \quad \ldots\ldots\ldots \quad (19);$$

and from the first

$$(\omega^2 + 2S - \frac{1}{\mu}RL + n^2)A + (2\omega n + \frac{1}{\mu}RM)B = 0 \quad \ldots\ldots \quad (20),$$

and $$\omega^2 - S - \frac{1}{\mu} RK = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21).$$

Eliminating A and B from these equations, we get

$$n^4 - \left\{3\omega^2 - 2S + \frac{1}{\mu} R(L - N)\right\} n^2 - 4\omega \frac{1}{\mu} RMn$$
$$+ (\omega^2 + 2S - \frac{1}{\mu} RL) \frac{1}{\mu} RN - \frac{1}{\mu^2} R^2 M^2 = 0 \quad . \quad . \quad . \quad . \quad (22),$$

a biquadratic equation to determine n.

For every *real* value of n there are terms in the expressions for ρ and σ of the form

$$A \cos(ms + nt + a).$$

For every *pure impossible* root of the form $\pm \sqrt{-1}\, n'$ there are terms of the forms

$$A\epsilon^{\pm n't} \cos(ms + a).$$

Although the negative exponential coefficient indicates a continually diminishing displacement which is consistent with stability, the positive value which necessarily accompanies it indicates a continually increasing disturbance, which would completely derange the system in course of time.

For every mixed root of the form $\pm \sqrt{-1}\, n' + n$, there are terms of the form

$$A\epsilon^{\pm n't} \cos(ms + nt + a).$$

If we take the positive exponential, we have a series of m waves travelling with velocity $\frac{n}{m}$ and increasing in amplitude with the coefficient $\epsilon^{+n't}$. The negative exponential gives us a series of m waves gradually dying away, but the negative exponential cannot exist without the possibility of the positive one having a finite coefficient, so that it is necessary for the stability of the motion that the four values of n be all real, and none of them either impossible quantities or the sums of possible and impossible quantities.

We have therefore to determine the relations among the quantities $KLMNRS$, that the equation

$$n^4 - \left\{S + \frac{1}{\mu} R(3K + L - N)\right\} n^2 - 4\omega \frac{1}{\mu} RMn$$
$$+ \left\{3S + \frac{1}{\mu} R(K - L)\right\} \frac{1}{\mu} RN - \frac{1}{\mu^2} R^2 M^2 = U = 0$$

may have four real roots.

7. In the first place, U is positive, when n is a large enough quantity, whether positive or negative.

It is also positive when $n = 0$, provided S be large, as it must be, compared with $\frac{1}{\mu} RL$, $\frac{1}{\mu} RM$ and $\frac{1}{\mu} RN$.

If we can now find a positive and a negative value of n for which U is negative, there must be four real values of n for which $U = 0$, and the four roots will be real.

Now if we put $n = \pm\sqrt{\frac{1}{2}}\sqrt{S}$,

$$U = -\frac{1}{4}S^2 + \frac{1}{2}\frac{1}{\mu}R(7N \pm 4\sqrt{2}M - L - 3K)S + \frac{1}{\mu^2}R^2(KN - LN - M^2),$$

which is negative if S be large compared to R.

So that a ring of satellites can always be rendered stable by increasing the mass of the central body and the angular velocity of the ring.

The values of L, M and N depend on m, the number of undulations in the ring. When $m = \frac{\mu}{2}$, the values of L and N will be at their maximum and $M = 0$. If we determine the relation between S and R in this case so that the system may be stable, the stability of the system for every other displacement will be secured.

8. To find the mass which must be given to the central body in order that a ring of satellites may permanently revolve round it.

We have seen that when the attraction of the central body is sufficiently great compared with the forces arising from the mutual action of the satellites, a permanent ring is possible. Now the forces between the satellites depend on the manner in which the displacement of each satellite takes place. The conception of a perfectly arbitrary displacement of all the satellites may be rendered manageable by separating it into a number of partial displacements depending on periodic functions. The motions arising from these small displacements will all take place independently, so that we have to consider only one at a time.

Of all these displacements, that which produces the greatest disturbing forces is that in which consecutive satellites are oppositely displaced, that is, when $m = \frac{\mu}{2}$, for then the nearest satellites are displaced so as to increase as much as possible the effects of the displacement of the satellite between them. If we make μ a large quantity, we shall have

$$\Sigma\frac{\sin^2 m\theta\cos^2\theta}{\sin^3\theta} = \frac{\mu^3}{\pi^3}(1 + 3^{-3} + 5^{-3} + \&c.) = \frac{\mu^3}{\pi^3}(1{\cdot}0518),$$

$$L = \frac{\mu^3}{\pi^3}{\cdot}5259, \qquad M = 0, \qquad N = 2L, \qquad K \text{ very small.}$$

Let $\frac{1}{\mu}RL = x$, then the equation of motion will be

$$n^4 - (S - x)n^2 + 2x(3S - x) = U = 0 \quad \ldots\ldots\ldots \quad (23).$$

The conditions of this equation having real roots are

$$S > x \quad \ldots\ldots\ldots\ldots \quad (24),$$

$$(S - x)^2 > 8x(3S - x) \quad \ldots\ldots\ldots \quad (25).$$

The last condition gives the equation

$$S^2 - 26Sx + 9x^2 > 0,$$

whence $$S > 25{\cdot}649x, \quad \text{or} \quad S < 0{\cdot}351x \quad \ldots\ldots\ldots \quad (26).$$

The last solution is inadmissible because S must be greater than x, so that the true condition is

$$S > 25{\cdot}649x,$$

$$> 25{\cdot}649 \frac{1}{\mu} R \frac{\mu^3}{\pi^3} {\cdot}5259,$$

$$S > {\cdot}4352\mu^2 R \quad \ldots\ldots\ldots\ldots \quad (27).$$

So that if there were 100 satellites in the ring, then

$$S > 4352\ R$$

is the condition which must be fulfilled in order that the motion arising from every conceivable displacement may be periodic.

If this condition be not fulfilled, and if S be not sufficient to render the motion perfectly stable, then although the motion depending upon long undulations may remain stable, the short undulations will increase in amplitude till some of the neighbouring satellites are brought into collision.

9. To determine the nature of the motion when the system of satellites is of small mass compared with the central body.

The equation for the determination of n is

$$U = n^4 - \left\{\omega^2 + \frac{1}{\mu} R(2K + L - N)\right\} n^2 - 4\omega \frac{1}{\mu} RMn$$

$$+ \left\{3\omega^2 - \frac{1}{\mu} R(2K + L)\right\} \frac{1}{\mu} RN - \frac{1}{\mu^2} R^2 M^2 = 0 \quad \ldots \quad (28).$$

When R is very small we may approximate to the values of n by assuming that two of them are nearly $\pm\omega$, and that the other two are small.

If we put $n = \pm\omega$,

$$U = -\frac{1}{\mu} R(2K + L \pm 4M - 4N)\,\omega^2 + \&\text{c.},$$

$$\frac{dU}{dn} = \pm 2\omega^3 + \&\text{c.}$$

Therefore the corrected values of n are

$$n = \pm\left\{\omega + \frac{1}{\mu\omega} R(2K + L - 4N)\right\} + \frac{1}{\mu\omega} RM \quad \ldots \quad (29).$$

The small values of n are nearly $\pm\sqrt{3\frac{1}{\mu}RN}$, correcting them in the same way, we find the approximate values

$$n = \pm\sqrt{3\frac{1}{\mu}RN} - 2\frac{1}{\mu\omega}RM \quad \ldots\ldots\ldots \quad (30).$$

The four values of n are therefore

$$\left.\begin{aligned} n_1 &= -\omega - \frac{1}{\mu\omega} R(2K + L - 4M - 4N) \\ n_2 &= -\sqrt{3\frac{1}{\mu} RN - \frac{2}{\mu\omega} RM} \\ n_3 &= +\sqrt{3\frac{1}{\mu} RN - \frac{2}{\mu\omega} RM} \\ n_4 &= +\omega + \frac{1}{\mu\omega} R(2K + L + 4M - 4N) \end{aligned}\right\} \quad \ldots\ldots \quad (31),$$

and the complete expression for ρ, so far as it depends on terms containing ms, is therefore

$$\rho = A_1 \cos(ms + n_1 t + \alpha_1) + A_2 \cos(ms + n_2 t + \alpha_2)$$
$$+ A_3 \cos(ms + n_3 t + \alpha_3) + A_4 \cos(ms + n_4 t + \alpha_4) \quad \ldots\ldots \quad (32),$$

and there will be other systems, of four terms each, for every value of m in the expansion of the original disturbance.

We are now able to determine the value of σ from equations (18), (20), by putting $\beta = \alpha$, and

$$B = -\frac{2\omega n + \frac{1}{\mu} RM}{n^2 + \frac{1}{\mu} RN} A \quad \ldots\ldots \quad (33).$$

So that for every term of ρ of the form

$$\rho = A\cos(ms + nt + \alpha) \quad \ldots\ldots \quad (34),$$

there is a corresponding term in σ,

$$\sigma = -\frac{2\omega n + \frac{1}{\mu} RM}{n^2 + \frac{1}{\mu} RN} A \sin(ms + nt + \alpha) \quad \ldots\ldots \quad (35).$$

10. Let us now fix our attention on the motion of a single satellite, and determine its motion by tracing the changes of ρ and σ while t varies and s is constant, and equal to the value of s corresponding to the satellite in question.

We must recollect that ρ and σ are measured outwards and forwards from an imaginary point revolving at distance 1 and velocity ω, so that the motions we consider are not the absolute motions of the satellite, but its motions relative to a point fixed in a revolving plane. This being understood, we may describe the motion as elliptic, the major axis being in the tangential direction, and the ratio of the axes being nearly $2\frac{\omega}{n}$, which is nearly 2 for n_1 and n_4 and is very large for n_2 and n_3.

The time of revolution is $\frac{2\pi}{n}$, or if we take a revolution of the ring as the unit of time, the time of a revolution of the satellite about its mean position is $\frac{\omega}{n}$.

The *direction* of revolution of the satellite about its mean position is in every case opposite to that of the motion of the ring.

11. The absolute motion of a satellite may be found from its motion relative to the ring by writing

$$r = 1 + \rho = 1 + A \cos(ms + nt + a),$$

$$\theta = \omega t + s + \sigma = \omega t + s - 2\frac{\omega}{n} A \sin(ms + nt + a).$$

When n is nearly equal to $\pm\omega$, the motion of each satellite in space is nearly elliptic. The excentricity is A, the longitude at epoch s, and the longitude when at the greatest distance from Saturn is for the negative value n_1

$$-\frac{1}{\mu\omega} R(2K + L - 4M - 4N)\, t + ms + a,$$

and for the positive value n_4

$$-\frac{1}{\mu\omega} R(2K + L + 4M - 4N)\, t - ms - a.$$

We must recollect that in all cases the quantity within brackets is negative, so that the major axis of the ellipse travels forwards in both cases. The chief difference between the two cases lies in the arrangement of the major axes of the ellipses of different satellites. In the first case as we pass from one satellite to the next in front the axes of the two ellipses lie in the same order. In the second case the particle in front has its major axis behind that of the other. In the cases in which n is small the radius vector of each satellite increases and diminishes during a periodic time of several revolutions. This gives rise to an inequality, in which the tangential displacement far exceeds the radial, as in the case of the *annual equation* of the Moon.

12. Let us next examine the condition of the ring of satellites at a given instant. We must therefore fix on a particular value of t and trace the changes of ρ and σ for different values of s.

From the expression for ρ we learn that the satellites form a wavy line, which is furthest from the centre when $(ms + nt + a)$ is a multiple of (2π), and nearest to the centre for intermediate values.

From the expression for σ we learn that the satellites are sometimes in advance and sometimes in the rear of their mean position, so that there are places where the satellites are crowded together, and others where they are drawn asunder. When n is positive, B is of the opposite sign to A, and the crowding of the satellites takes place when they are furthest from the centre. When n is negative, the satellites are separated most when furthest from the centre, and crowded together when they approach it.

The form of the ring at any instant is therefore that of a string of beads forming a re-entering curve, nearly circular, but with a small variation of distance from the centre recurring m times, and forming m regular waves of transverse displacement at equal intervals round the circle. Besides these, there are waves of condensation and rarefaction, the effect of longitudinal displacement. When n is positive the points of greatest distance from the centre are points of greatest condensation, and when n is negative they are points of greatest rarefaction.

13. We have next to determine the velocity with which these waves of disturbance are propagated round the ring. We fixed our attention on a particular satellite by making s constant, and on a particular instant by making t constant, and thus we determined the motion of a satellite and the form of the ring. We must now fix our attention on a *phase* of the motion, and this we do by making ρ or σ constant. This implies

$$ms + nt + \alpha = \text{constant},$$

$$\frac{ds}{dt} = -\frac{n}{m}.$$

So that the particular phase of the disturbance travels round the ring with an angular velocity $= -\frac{n}{m}$ relative to the ring itself. Now the ring is revolving in space with the velocity ω, so that the angular velocity of the wave in space is

$$\varpi = \omega - \frac{n}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36).$$

Thus each satellite moves in an ellipse, while the general aspect of the ring is that of a curve of m waves revolving with velocity ϖ. This, however, is only the part of the whole motion, which depends on a single term of the solution. In order to understand the general solution we must shew how to determine the whole motion from the state of the ring at a given instant.

14. *Given the position and motion of every satellite at any one time, to calculate the position and motion of every satellite at any other time, provided that the condition of stability is fulfilled.*

The position of any satellite may be denoted by the values of ρ and σ for that satellite, and its velocity and direction of motion are then indicated by the values of $\frac{d\rho}{dt}$ and $\frac{d\sigma}{dt}$ at the given instant.

These four quantities may have for each satellite any four arbitrary values, as the position and motion of each satellite are independent of the rest, at the beginning of the motion.

Each of these quantities is therefore a perfectly arbitrary function of s, the mean angular position of the satellite in the ring.

But any function of s from $s = 0$ to $s = 2\pi$, however arbitrary or discontinuous, can be expanded in a series of terms of the form $A \cos(s + \alpha) + A' \cos(2s + \alpha') +$ &c. See § 3.

Let each of the four quantities ρ, $\frac{d\rho}{dt}$, σ, $\frac{d\sigma}{dt}$ be expressed in terms of such a series, and let the terms in each involving ms be

$$\rho = E \cos(ms + e) \quad \ldots \quad (37),$$

$$\frac{d\rho}{dt} = F \cos(ms + f) \quad \ldots \quad (38),$$

$$\sigma = G \cos(ms + g) \quad \ldots \quad (39),$$

$$\frac{d\sigma}{dt} = H \cos(ms + h) \quad \ldots \quad (40).$$

These are the parts of the values of each of the four quantities which are capable of being expressed in the form of periodic functions of ms. It is evident that the eight quantities E, F, G, H, e, f, g, h, are all independent and arbitrary.

The next operation is to find the values of L, M, N, belonging to disturbances in the ring whose index is m, [see equation (8)], to introduce these values into equation (28), and to determine the four values of n, $(n_1\ n_2\ n_3\ n_4)$.

This being done, the expression for ρ is that given in equation (32), which contains eight arbitrary quantities $(A_1\ A_2\ A_3\ A_4\ \alpha_1\ \alpha_2\ \alpha_3\ \alpha_4)$.

Giving t its original value in this expression, and equating it to $E \cos(ms + e)$, we get an equation which is equivalent to two. For, putting $ms = 0$, we have

$$A_1 \cos\alpha_1 + A_2 \cos\alpha_2 + A_3 \cos\alpha_3 + A_4 \cos\alpha_4 = E \cos e \quad \ldots \quad (41).$$

And putting $ms = \frac{\pi}{2}$, we have another equation

$$A_1 \sin\alpha_1 + A_2 \sin\alpha_2 + A_3 \sin\alpha_3 + A_4 \sin\alpha_4 = E \sin e \quad \ldots \quad (42).$$

Differentiating (32) with respect to t, we get two other equations

$$-A_1 n_1 \sin\alpha - \text{\&c.} = F \cos f \quad \ldots \quad (43),$$

$$A_1 n_1 \cos\alpha + \text{\&c.} = F \sin f \quad \ldots \quad (44).$$

Bearing in mind that $B_1\ B_2$ &c. are connected with $A_1\ A_2$ &c. by equation (33), and that B is therefore proportional to A, we may write $B = A\beta$, where

$$\beta = -\frac{2\omega n + \frac{1}{\mu} RM}{n^2 + \frac{1}{\mu} NR},$$

β being thus a function of n and a known quantity.

The value of σ then becomes at the epoch

$$\sigma = A_1\beta_1 \sin(ms + \alpha_1) + \&c. = G\cos(ms + g),$$

from which we obtain the two equations

$$A_1\beta_1 \sin\alpha_1 + \&c. = G\cos g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45),$$

$$A_1\beta_1 \cos\alpha_1 + \&c. = -G\sin g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46).$$

Differentiating with respect to t, we get the remaining equations

$$A_1\beta_1 n_1 \cos\alpha_1 + \&c. = H\cos h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47),$$

$$A_1\beta_1 n_1 \sin\alpha_1 + \&c. = H\sin h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48).$$

We have thus found eight equations to determine the eight quantities A_1 &c. and α_1 &c: To solve them, we may take the four in which $A_1 \cos \alpha_1$ &c. occur, and treat them as simple equations, so as to find $A_1 \cos \alpha_1$ &c. Then taking those in which $A_1 \sin \alpha_1$ &c. occur, and determining the values of those quantities, we can easily deduce the values of A_1 and of α_1 &c., from these.

We now know the amplitude and phase of each of the four waves whose index is m. All other systems of waves belonging to any other index must be treated in the same way, and since the original disturbance, however arbitrary, can be broken up into periodic functions of the form of equations (37—40), our solution is perfectly general, and applicable to every possible disturbance of a ring fulfilling the condition of stability (27).

15. We come next to consider the effect of an external disturbing force, due either to the irregularities of the planet, the attraction of satellites, or the motion of waves in other rings.

All disturbing forces of this kind may be expressed in series of which the general term is

$$A \cos(vt + ms + \alpha),$$

where v is an angular velocity and m a whole number.

Let $P\cos(ms + vt + p)$ be the central part of the force, acting inwards, and $Q\sin(ms + vt + q)$ the tangential part, acting forwards. Let $\rho = A\cos(ms + vt + a)$ and $\sigma = B\sin(ms + vt + \beta)$, be the terms of ρ and σ which depend on the external disturbing force. These will simply be added to the terms depending on the original disturbance which we have already investigated, so that the complete expressions for ρ and σ will be as general as before. In consequence of the additional forces and displacements, we must add to equations (16) and (17), respectively, the following terms:

$$\left\{3\omega^2 - \frac{1}{\mu}R(2K + L) + v^2\right\} A\cos(ms + vt + a)$$
$$+ \left(2\omega v + \frac{1}{\mu}RM\right) B\cos(ms + vt + \beta) - P\cos(ms + vt + p) = 0 \quad . \quad . \quad . \quad (49).$$

$$\left(2\omega v + \frac{1}{\mu}RM\right) A\sin(ms + vt + a)$$
$$+ \left(v^2 + \frac{1}{\mu}RN\right) B\sin(ms + vt + \beta) + Q\sin(ms + vt + q) = 0 \quad . \quad . \quad . \quad (50).$$

Making $(ms + vt) = 0$ in the first equation and $\frac{\pi}{2}$ in the second,

$$\left\{3\omega^2 - \frac{1}{\mu} R(2K+L) + v^2\right\} A\cos\alpha + \left(2\omega v + \frac{1}{\mu} RM\right) B\cos\beta - P\cos p = 0 \quad . \quad . \quad . \quad (51).$$

$$\left(2\omega v + \frac{1}{\mu} RM\right) A\cos\alpha + \left(v^2 + \frac{1}{\mu} RN\right) B\cos\beta + Q\cos q = 0 \quad . \quad . \quad . \quad (52).$$

Then if we put

$$U' = v^4 - \left\{\omega^2 + \frac{1}{\mu} R(2K + L - N)\right\} v^2 - 4\frac{\omega}{\mu} RMv$$
$$+ \left\{3\omega^2 - \frac{1}{\mu} R(2K+L)\right\} \frac{1}{\mu} RN - \frac{1}{\mu^2} R^2 M^2 \quad . \quad . \quad . \quad . \quad (53),$$

we shall find the value of $A\cos\alpha$ and $B\cos\beta$;

$$A\cos\alpha = \frac{v^2 + \frac{1}{\mu} RN}{U'} P\cos p + \frac{2\omega v + \frac{1}{\mu} RM}{U'} Q\cos q \quad . \quad . \quad . \quad (54).$$

$$B\cos\beta = -\frac{2\omega v + \frac{1}{\mu} RM}{U'} P\cos p - \frac{v^2 + 3\omega^2 - \frac{1}{\mu} R(K+L)}{U'} Q\cos q \quad . \quad . \quad (55).$$

Substituting sines for cosines in equations (51) (52), we may find the values of $A\sin\alpha$ and $B\sin\beta$.

Now U' is precisely the same function of v that U is of n, so that if v coincides with one of the four values of n, U' will vanish, the coefficients A and B will become infinite, and the ring will be destroyed. The disturbing force is supposed to arise from a revolving body, or an undulation of any kind which has an angular velocity $-\frac{v}{m}$ relatively to the ring, and therefore an absolute angular velocity $= \omega - \frac{v}{m}$.

If then the absolute angular velocity of the disturbing body is exactly or nearly equal to the absolute angular velocity of any of the free waves of the ring, that wave will increase till the ring be destroyed.

The velocities of the free waves are nearly

$$\omega\left(1 + \frac{1}{m}\right),\ \omega + \frac{1}{m}\sqrt{3\frac{1}{\mu}RN},\ \omega - \frac{1}{m}\sqrt{3\frac{1}{\mu}RN}, \text{ and } \omega\left(1 - \frac{1}{m}\right) \quad . \quad . \quad . \quad (56).$$

When the angular velocity of the disturbing body is greater than that of the first wave, between those of the second and third, or less than that of the fourth, U' is positive. When it is between the first and second, or between the third and fourth, U' is negative.

Let us now simplify our conception of the disturbance by attending to the central force only, and let us put $p = 0$, so that P is a maximum when $ms + vt$ is a multiple of 2π. We find in this case $\alpha = 0$, and $\beta = 0$. Also

$$A = \frac{v^2 + \frac{1}{\mu} RN}{U'} P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57),$$

$$B = -\frac{2\omega v + \frac{1}{\mu} RM}{U'} P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58).$$

When U is positive, A will be of the same sign as P, that is, the parts of the ring will be furthest from the centre where the disturbing force towards the centre is greatest. When U is negative, the contrary will be the case.

When v is positive, B will be of the opposite sign to A, and the parts of the ring furthest from the centre will be most crowded. When v is negative the contrary will be the case.

Let us now attend only to the tangential force, and let us put $q = 0$. We find in this case also $\alpha = 0$, $\beta = 0$,

$$A = \frac{2\omega v + \frac{1}{\mu} RM}{U'} Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59),$$

$$B = -\frac{v^2 + 3\omega^2 - \frac{1}{\mu} R(K + L)}{U'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60).$$

The tangential displacement is here in the same or in the opposite direction to the tangential force, according as U' is negative or positive. The crowding of satellites is at the points farthest from or nearest to Saturn according as v is positive or negative.

16. The effect of any disturbing force is to be determined in the following manner. The disturbing force, whether radial or tangential, acting on the ring may be conceived to vary from one satellite to another, and to be different at different times. It is therefore a perfectly arbitrary function of s and t.

Let Fourier's method be applied to the general disturbing force so as to divide it up into terms depending on periodic functions of s, so that each term is of the form $F(t) \cos(ms + \alpha)$, where the function of t is still perfectly arbitrary.

But it appears from the general theory of the permanent motions of the heavenly bodies that they may all be expressed by periodic functions of t arranged in series. Let vt be the argument of one of these terms, then the corresponding term of the disturbance will be of the form

$$P \cos(ms + vt + \alpha).$$

This term of the disturbing force indicates an alternately positive and negative action, disposed in m waves round the ring, completing its period relatively to each particle in the time $\frac{2\pi}{v}$, and travelling as a wave among the particles with an angular velocity $-\frac{v}{m}$, the angular velocity relative to fixed space being of course $\omega - \frac{v}{m}$. The whole disturbing force may be split up into terms of this kind.

17. Each of these elementary disturbances will produce its own wave in the ring, independent of those which belong to the ring itself. This new wave, due to external disturbance, and following different laws from the natural waves of the ring, is called the *forced wave.* The angular velocity of the forced wave is the same as that of the disturbing force, and its maxima and minima coincide with those of the force, but the extent of the disturbance and its direction depend on the comparative velocities of the forced wave and the four natural waves.

When the velocity of the forced wave lies between the velocities of the two middle free waves, or is greater than that of the swiftest, or less than that of the slowest, then the radial displacement due to a radial disturbing force is in the same direction as the force, but the tangential displacement due to a tangential disturbing force is in the opposite direction to the force.

The radial force therefore in this case produces a *positive forced wave*, and the tangential force a *negative forced wave.*

When the velocity of the forced wave is either between the velocities of the first and second free waves, or between those of the third and fourth, then the radial disturbance produces a forced wave in the contrary direction to that in which it acts, or a negative wave, and the tangential force produces a positive wave.

The coefficient of the forced wave changes sign whenever its velocity passes through the value of any of the velocities of the free waves, but it does so by becoming infinite, and not by vanishing, so that when the angular velocity very nearly coincides with that of a free wave, the forced wave becomes very great, and if the velocity of the disturbing force were made exactly equal to that of a free wave, the coefficient of the forced wave would become infinite. In such a case we should have to readjust our approximations, and to find whether such a coincidence might involve a physical impossibility.

The forced wave which we have just investigated is that which would maintain itself in the ring, supposing that it had been set agoing at the commencement of the motion. It is in fact the form of dynamical equilibrium of the ring under the influence of the given forces. In order to find the actual motion of the ring we must combine this forced wave with all the free waves, which go on independently of it, and in this way the solution of the problem becomes perfectly complete, and we can determine the whole motion under any given initial circumstances, as we did in the case where no disturbing force acted.

For instance, if the ring were perfectly uniform and circular at the instant when the disturbing force began to act, we should have to combine with the constant forced wave a system of four free waves so disposed, that at the given epoch, the displacements due to them should exactly neutralize those due to the forced wave. By the combined effect of these four free waves and the forced one the whole motion of the ring would be accounted for, beginning from its undisturbed state.

The disturbances which are of most importance in the theory of Saturn's rings are those which are produced in one ring by the action of attractive forces arising from waves belonging to another ring.

The effect of this kind of action is to produce in each ring, besides its own four free waves, four forced waves corresponding to the free waves of the other ring. There will thus be eight waves in each ring, and the corresponding waves in the two rings will act and react on each other, so that, strictly speaking, every one of the waves will be in some measure a forced wave, although the system of eight waves will be the free motion of the two rings taken together. The theory of the mutual disturbance and combined motion of two concentric rings of satellites requires special consideration.

18. On the motion of a ring of satellites when the conditions of stability are not fulfilled.

We have hitherto been occupied with the case of a ring of satellites, the stability of which was ensured by the smallness of mass of the satellites compared with that of the central body. We have seen that the statically unstable condition of each satellite between its two immediate neighbours may be compensated by the dynamical effect of its revolution round the planet, and a planet of sufficient mass can not only direct the motion of such satellites round its own body, but can likewise exercise an influence over their relations to each other, so as to overrule their natural tendency to crowd together, and distribute and preserve them in the form of a ring.

We have traced the motion of each satellite, the general shape of the disturbed ring, and the motion of the various waves of disturbance round the ring, and determined the laws both of the natural or free waves of the ring, and of the forced waves, due to extraneous disturbing forces.

We have now to consider the cases in which such a permanent motion of the ring is impossible, and to determine the mode in which a ring, originally regular, will break up, in the different cases of instability.

The equation from which we deduce the conditions of stability is—

$$U = n^4 - \left\{\omega^2 + \frac{1}{\mu} R\,(2K + L - N)\right\} n^2 - 4\omega \frac{1}{\mu} RMn + \left\{3\omega^2 - \frac{1}{\mu} R\,(2K + L) \frac{1}{\mu} RN\right\} - \frac{1}{\mu^2} R^2M^2 = 0.$$

The quantity, which, in the critical cases, determines the nature of the roots of this equation, is N. The quantity M in the third term is always small compared with L and N when m is large, that is, in the case of the dangerous short waves. We may therefore begin our study of the critical cases by leaving out the third term. The equation then becomes a quadratic in n^2, and in order that all the values of n may be real, both values of n^2 must be real and positive.

The condition of the values of n^2 being real is

$$\omega^4 + \omega^2 \frac{1}{\mu} R\,(4K + 2L - 14N) + \frac{1}{\mu^2} R^2\,(2K + L + N)^2 > 0 \quad . \quad . \quad . \quad . \quad (61),$$

which shows that ω^2 must either be about 14 times at least smaller, or about 14 times at least greater, than quantities like $\frac{1}{\mu} RN$.

That both values of n^2 may be positive, we must have,

$$\left.\begin{array}{r} \omega^2 + \frac{1}{\mu} R\,(2K + L - N) > 0 \\ \left\{3\omega^2 - \frac{1}{\mu} R\,(2K + L)\right\} \frac{1}{\mu} RN > 0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62).$$

We must therefore take the larger values of ω^2, and also add the condition that N be positive.

We may therefore state roughly, that, to ensure stability, $\frac{RN}{\mu}$, the coefficient of tangential attraction, must lie between zero and $\frac{1}{14}\omega^2$. If the quantity be negative, the two *small* values of n will become *pure impossible* quantities. If it exceed $\frac{1}{14}\omega^2$, *all* the values of n will take the form of mixed impossible quantities.

If we write x for $\frac{1}{\mu} RN$, and omit the other disturbing forces, the equation becomes

$$U = n^4 - (\omega^2 - x)\, n^2 + 3\omega^2 x = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63),$$

whence

$$n^2 = \tfrac{1}{2}\,(\omega^2 - x) \pm \tfrac{1}{2}\sqrt{\omega^4 - 14\omega^2 x + x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64).$$

If x be small, two of the values of n are nearly $\pm\,\omega$, and the others are small quantities, real when x is positive and impossible when x is negative.

If x be greater than $(7 - \sqrt{48})\,\omega^2$, or $\frac{\omega^2}{14}$ nearly, the term under the radical becomes negative, and the value of n becomes

$$n = \pm \tfrac{1}{2}\sqrt{\sqrt{12\omega^2 x} + \omega^2 - x} \pm \tfrac{1}{2}\sqrt{-1}\sqrt{\sqrt{12\omega^2 x} - \omega^2 + x} \quad . \quad . \quad . \quad (65),$$

where one of the terms is a real quantity, and the other impossible. Every solution may be put under the form

$$n = p \pm \sqrt{-1}\, q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66),$$

where $q = 0$ for the case of stability, $p = 0$ for the pure impossible roots, and p and q finite for the mixed roots.

Let us now adopt this general solution of the equation for n, and determine its mechanical significance by substituting for the impossible circular functions their equivalent real exponential functions.

Substituting the general value of n in equations (34), (35),

$$\rho = A\left[\cos\left\{ms + (p + \sqrt{-1}q)\,t + a\right\} + \cos\left\{ms + (p - \sqrt{-1}q)\,t + a\right\}\right] \quad . \quad (67),$$

$$\left.\begin{array}{l} \sigma = -A\,\dfrac{2\omega\,(p + \sqrt{-1}\,q)}{(p + \sqrt{-1}q)^2 + x} \sin\left\{ms + (p + \sqrt{-1}\,q)\,t + a\right\} \\ \qquad -A\,\dfrac{2\omega\,(p - \sqrt{-1}\,q)}{(p - \sqrt{-1}\,q)^2 + x} \sin\left\{ms + (p - \sqrt{-1}\,q)\,t + a\right\} \end{array}\right\} \quad . \quad . \quad (68).$$

Introducing the exponential notation, these values become

$$\rho = A\,(\epsilon^{qt} + \epsilon^{-qt}) \cos (ms + pt + \alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69),$$

$$\sigma = -\frac{2\omega A}{(p^2+q^2)^2 + 2(p^2-q^2)x + x^2} \left\{ \begin{array}{l} p(p^2+q^2+x)\,(\epsilon^{qt}+\epsilon^{-qt}) \sin (ms+pt+\alpha) \\ + q\,(p^2+q^2-x)\,(\epsilon^{qt}-\epsilon^{-qt}) \cos (ms+pt+\alpha) \end{array} \right\}. \quad (70).$$

We have now obtained a solution free from impossible quantities, and applicable to every case.

When $q = 0$, the case becomes that of real roots, which we have already discussed. When $p = 0$, we have the case of pure impossible roots arising from the negative values of n^2. The solutions corresponding to these roots are

$$\rho = A\,(\epsilon^{qt} + \epsilon^{-qt}) \cos (ms + \alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71).$$

$$\sigma = -\frac{2\omega q A}{q^2 - x}\,(\epsilon^{qt} - \epsilon^{-qt}) \cos (ms + \alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72).$$

The part of the coefficient depending on ϵ^{-qt} diminishes indefinitely as the time increases, and produces no marked effect. The other part, depending on ϵ^{qt}, increases in a geometrical proportion as the time increases arithmetically, and so breaks up the ring. In the case of x being a small negative quantity, q^2 is nearly $3x$, so that the coefficient of σ becomes

$$-3\frac{\omega}{q}A.$$

It appears therefore that the motion of each particle is either outwards and backwards or inwards and forwards, but that the tangential part of the motion greatly exceeds the normal part.

It may seem paradoxical that a tangential force, acting *towards* a position of equilibrium, should produce instability, while a small tangential force *from* that position ensures stability, but it is easy to trace the destructive tendency of this apparently conservative force.

Suppose a particle slightly in front of a crowded part of the ring, then if x is negative there will be a tangential force pushing it forwards, and this force will cause its distance from the planet to increase, its angular velocity to diminish, and the particle itself to fall back on the crowded part, thereby increasing the irregularity of the ring, till the whole ring is broken up. In the same way it may be shown that a particle *behind* a crowded part will be pushed into it. The only force which could preserve the ring from the effect of this action, is one which would prevent the particle from receding from the planet under the influence of the tangential force, or at least prevent the diminution of angular velocity. The transversal force of attraction of the ring is of this kind, and acts in the right direction, but it can never be of sufficient magnitude to have the required effect. In fact the thing to be done is to render the least term of the equation in n^2 positive when N is negative, which requires

$$\frac{1}{\mu} R(2K + L) > 3\omega^2,$$

and this condition is quite inconsistent with any constitution of the ring which fulfils the other condition of stability which we shall arrive at presently.

We may observe that the waves belonging to the two real values of n, $\pm\omega$, must be conceived to be travelling round the ring during the whole time of its breaking up, and conducting themselves like ordinary waves, till the excessive irregularities of the ring become inconsistent with their uniform propagation.

The irregularities which depend on the exponential solutions do not travel round the ring by propagation among the satellites, but remain among the same satellites which first began to move irregularly.

We have seen the fate of the ring when x is negative. When x is small we have two small and two large values of n, which indicate regular waves, as we have already shown. As x increases, the small values of n increase, and the large values diminish, till they meet and form a pair of positive and a pair of negative equal roots, having values nearly $\pm\cdot 68\omega$. When x becomes greater than about $\frac{1}{14}\omega^2$, then all the values of n become impossible, of the form $p+\sqrt{-1}\,q$, q being small when x first begins to exceed its limits, and p being nearly $\pm\cdot 68\omega$.

The values of ρ and σ indicate periodic inequalities having the period $\frac{2\pi}{p}$, but increasing in amplitude at a rate depending on the exponential ϵ^{qt}. At the beginning of the motion the oscillations of the particles are in ellipses as in the case of stability, having the ratio of the axes about 1 in the normal direction to 3 in the tangential direction. As the motion continues, these ellipses increase in magnitude, and another motion depending on the second term of σ is combined with the former, so as to increase the ellipticity of the oscillations and to turn the major axis into an inclined position, so that its fore end points a little inwards, and its hinder end a little outwards. The oscillations of each particle round its mean position are therefore in ellipses, of which both axes increase continually while the eccentricity increases, and the major axis becomes slightly inclined to the tangent, and this goes on till the ring is destroyed. In the mean time the irregularities of the ring do not remain among the same set of particles as in the former case, but travel round the ring with a relative angular velocity $-\frac{p}{m}$. Of these waves there are four, two travelling forwards among the satellites, and two travelling backwards. One of each of these pairs depends on a negative value of q, and consists of a wave whose amplitude continually decreases. The other depends on a positive value of q, and is the destructive wave whose character we have just described.

19. We have taken the case of a ring composed of equal satellites, as that with which we may compare other cases in which the ring is constructed of loose materials differently arranged.

In the first place let us consider what will be the conditions of a ring composed of satellites of unequal mass. We shall find that the motion is of the same kind as when the satellites are equal.

For by arranging the satellites so that the smaller satellites are closer together than the larger ones, we may form a ring which will revolve uniformly about Saturn, the resultant force on each satellite being just sufficient to keep it in its orbit.

To determine the stability of this kind of motion, we must calculate the disturbing forces due to any given displacement of the ring. This calculation will be more complicated than in the former case, but will lead to results of the same general character. Placing these forces in the equations of motion, we shall find a solution of the same general character as in the former case, only instead of regular waves of displacement travelling round the ring, each wave will be split and reflected when it comes to irregularities in the chain of satellites. But if the condition of stability for every kind of wave be fulfilled, the motion of each satellite will consist of small oscillations about its position of dynamical equilibrium, and thus, on the whole, the ring will of itself assume the arrangement necessary for the continuance of its motion, if it be originally in a state not very different from that of equilibrium.

20. We now pass to the case of a ring of an entirely different construction. It is possible to conceive of a quantity of matter, either solid or liquid, not collected into a continuous mass, but scattered thinly over a great extent of space, and having its motion regulated by the gravitation of its parts to each other, or towards some dominant body. A shower of rain, hail, or cinders is a familiar illustration of a number of unconnected particles in motion; the visible stars, the milky way, and the resolved nebulæ, give us instances of a similar scattering of bodies on a larger scale. In the terrestrial instances we see the motion plainly, but it is governed by the attraction of the earth, and retarded by the resistance of the air, so that the mutual attraction of the parts is completely masked. In the celestial cases the distances are so enormous, and the time during which they have been observed so short, that we can perceive no motion at all. Still we are perfectly able to conceive of a collection of particles of small size compared with the distances between them, acting upon one another only by the attraction of gravitation, and revolving round a central body. The average density of such a system may be smaller than that of the rarest gas, while the particles themselves may be of great density; and the appearance from a distance will be that of a cloud of vapour, with this difference, that as the space between the particles is empty, the rays of light will pass through the system without being refracted, as they would have been if the system had been gaseous.

Such a system will have an *average density* which may be greater in some places than others. The resultant attraction will be towards places of greater average density, and thus the density of those places will be increased so as to increase the irregularities of density. The system will therefore be statically unstable, and nothing but motion of some kind can prevent the particles from forming agglomerations, and these uniting, till all are reduced to one solid mass.

We have already seen how dynamical stability can exist where there is statical instability in the case of a row of particles revolving round a central body. Let us now conceive a cloud of particles forming a ring of nearly uniform density revolving about a central body. There will be a primary effect of inequalities in density tending to draw particles towards the denser parts of the ring, and this will elicit a secondary effect, due to the motion of revolution, tending in the contrary direction, so as to restore the rings to uniformity. The relative magnitude of these two opposing forces determines the destruction or preservation of the ring.

To calculate these effects we must begin with the statical problem:—To determine the forces arising from the given displacements of the ring.

The longitudinal force arising from longitudinal displacements is that which has most effect in determining the stability of the ring. In order to estimate its limiting value we shall solve a problem of a simpler form.

21. An infinite mass, originally of uniform density k, has its particles displaced by a quantity ξ parallel to the axis of x, so that $\xi = A \cos mx$, to determine the attraction on each particle due to this displacement.

The density at any point will differ from the original density by a quantity k', so that

$$(k + k')(dx + d\xi) = kdx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73).$$

$$k' = -k\frac{d\xi}{dx} = Akm \sin mx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74).$$

The potential at any point will be $V + V'$, where V is the original potential, and V' depends on the displacement only, so that

$$\frac{d^2V'}{dx^2} + \frac{d^2V'}{dy^2} + \frac{d^2V'}{dz^2} + 4\pi k' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75).$$

Now V' is a function of x only, and therefore,

$$V' = 4\pi Ak\frac{1}{m}\sin mx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76),$$

and the longitudinal force is found by differentiating V' with respect to x.

$$X = \frac{dV'}{dx} = 4\pi kA \cos mx = 4\pi k\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77).$$

Now let us suppose this mass not of infinite extent, but of finite section parallel to the plane of yz. This change amounts to cutting off all portions of the mass beyond a certain boundary. Now the effect of the portion so cut off upon the longitudinal force depends on the value of m. When m is large, so that the wave-length is small, the effect of the external portion is insensible, so that the longitudinal force due to short waves is not diminished by cutting off a great portion of the mass.

22. Applying this result to the case of a ring, and putting s for x, and σ for ξ, we have

$$\sigma = A\cos ms, \text{ and } T = 4\pi kA\cos ms,$$

so that

$$\frac{1}{\mu}RN = 4\pi k,$$

when m is very large, and this is the greatest value of N.

The value of L has little effect on the condition of stability. If L and M are both neglected, that condition is $\omega^2 > 27{\cdot}856\,(2\pi k)$ . . . . (78),

and if L be as much as $\frac{1}{2}N$, then $\omega^2 > 25{\cdot}649\,(2\pi k)$ . . . . (79),

so that it is not important whether we calculate the value of L or not.

The condition of stability is, that the average density must not exceed a certain value. Let us ascertain the relation between the maximum density of the ring and that of the planet.

Let b be the radius of the planet, that of the ring being unity, then the mass of Saturn is $\frac{4}{3}\pi b^3 k' = \omega^2$ if k' be the density of the planet. If we assume that the radius of the ring is twice that of the planet, as Laplace has done, then $b = \frac{1}{2}$ and

$$\frac{k'}{k} = 334{\cdot}2 \text{ to } 307{\cdot}7 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (80),$$

so that the density of the ring cannot exceed $\frac{1}{300}$ of that of the planet. Now Laplace has shown that if the outer and inner parts of the ring have the same angular velocity, the ring will not hold together if the ratio of the density of the planet to that of the ring exceeds 1·3, so that in the first place, our ring cannot have uniform angular velocity, and in the second place, Laplace's ring cannot preserve its form, if it is composed of loose materials acting on each other only by the attraction of gravitation, and moving with the same angular velocity throughout.

23. On the forces arising from inequalities of thickness in a thin stratum of fluid of indefinite extent.

The forces which act on any portion of a continuous fluid are of two kinds, the pressures of contiguous portions of fluid, and the attractions of all portions of the fluid whether near or distant. In the case of a thin stratum of fluid, not acted on by any external forces, the pressures are due mainly to the component of the attraction which is perpendicular to the plane of the stratum. It is easy to show that a fluid acted on by such a force will tend to assume a position of equilibrium, in which its free surface is plane; and that any irregularities will tend to equalise themselves, so that the plane surface will be one of stable equilibrium.

It is also evident, that if we consider only that part of the attraction which is parallel to the plane of the stratum, we shall find it always directed towards the thicker parts, so that the effect of this force is to draw the fluid from thinner to thicker parts, and so to increase irregularities and destroy equilibrium.

The normal attraction therefore tends to preserve the stability of equilibrium, while the tangential attraction tends to render equilibrium unstable.

According to the nature of the irregularities one or other of these forces will prevail, so that if the extent of the irregularities is small, the normal forces will ensure stability, while, if the inequalities cover much space, the tangential forces will render equilibrium unstable, and break up the stratum into beads.

To fix our ideas, let us conceive the irregularities of the stratum split up into the form of a number of systems of waves superposed on one another, then, by what we have just said, it appears, that very short waves will disappear of themselves, and be consistent with stability, while very long waves will tend to increase in height, and will destroy the form of the stratum.

In order to determine the law according to which these opposite effects take place, we must subject the case to mathematical investigation.

Let us suppose the fluid incompressible, and of the density k, and let it be originally contained between two parallel planes, at distances $+c$ and $-c$ from that of (xy), and extending to infinity. Let us next conceive a series of imaginary planes, parallel to the plane of (yz), to be plunged into the fluid stratum at infinitesimal distances from one another, so as to divide the fluid into imaginary slices perpendicular to the plane of the stratum.

Next let these planes be displaced parallel to the axis of x according to this law—that if x be the original distance of the plane from the origin, and ξ its displacement in the direction of x,

$$\xi = A \cos mx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81).$$

According to this law of displacement, certain alterations will take place in the distances between consecutive planes; but since the fluid is incompressible, and of indefinite extent in the direction of y, the change of dimension must occur in the direction of z. The original thickness of the stratum was $2c$. Let its thickness at any point after displacement be $2c + 2\zeta$, then we must have

$$(2c + 2\zeta)\left(1 + \frac{d\xi}{dx}\right) = 2c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82),$$

or

$$\zeta = -c\frac{d\xi}{dx} = cmA \sin mx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83).$$

Let us assume that the increase of thickness 2ζ is due to an increase of ζ at each surface; this is necessary for the equilibrium of the fluid between the imaginary planes.

We have now produced artificially, by means of these planes, a system of waves of longitudinal displacement whose length is $\frac{2\pi}{m}$ and amplitude A; and we have found that this has produced a system of waves of normal displacement on each surface, having the same length, with a height $= cmA$.

In order to determine the forces arising from these displacements, we must, in the first place, determine the potential function at any point of space, and this depends partly on the state of the fluid before displacement, and partly on the displacement itself. We have, in all cases—

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = -4\pi\rho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84).$$

Within the fluid, $\rho = k$; beyond it, $\rho = 0$.

Before displacement, the equation is reduced to

$$\frac{d^2 V}{dz^2} = -4\pi\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85).$$

Instead of assuming $V = 0$ at infinity, we shall assume $V = 0$ at the origin, and since in this case all is symmetrical, we have

$$\left.\begin{array}{lll} \text{within the fluid} & V_1 = -2\pi k z^2, & \dfrac{dV}{dz} = -4\pi k z \\ \text{at the bounding planes} & V = -2\pi k c^2 & \dfrac{dV}{dz} = \mp 4\pi k c \\ \text{beyond them} & V_2 = 2\pi k c\,(\mp 2z - c) & \dfrac{dV}{dz} = \mp 4\pi k c \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (86);$$

the upper sign being understood to refer to the boundary at distance $+c$, and the lower to the boundary at distance $-c$ from the origin.

Having ascertained the potential of the undisturbed stratum, we find that of the disturbance by calculating the effect of a stratum of density k and thickness ζ, spread over each surface according to the law of thickness already found. By supposing the coefficient A small enough, (as we may do in calculating the displacements on which stability depends,) we may diminish the absolute thickness indefinitely, and reduce the case to that of a mere "superficial density," such as is treated of in the theory of electricity. We have here, too, to regard some parts as of *negative* density; but we must recollect that we are dealing with the *difference* between a disturbed and an undisturbed system, which may be positive or negative, though no real mass can be negative.

Let us for an instant conceive only one of these surfaces to exist, and let us transfer the origin to it. Then the law of thickness is

$$\zeta = mcA \sin mx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83),$$

and we know that the normal component of attraction at the surface is the same as if the thickness had been uniform throughout, so that

$$\frac{dV}{dz} = -2\pi k\zeta,$$

on the positive side of the surface.

Also, the solution of the equation

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dz^2} = 0,$$

consists of a series of terms of the form $C\epsilon^{iz} \sin ix$

Of these the only one with which we have to do is that in which $i = -m$. Applying the condition as to the normal force at the surface, we get

$$V = 2\pi kc\epsilon^{-mz}A \sin mx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

for the potential on the positive side of the surface, and

$$V = 2\pi kc\epsilon^{mz}A \sin mx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88).$$

on the negative side.

Calculating the potentials of a pair of such surfaces at distances $+c$ and $-c$ from the plane of xy, and calling V' the sum of their potentials, we have for the space between these planes

$$\left.\begin{aligned} V_1' &= 2\pi kcA \sin mx\epsilon^{-mc}(\epsilon^{mz} + \epsilon^{-mz}) \\ \text{beyond them} \qquad V_2' &= 2\pi kcA \sin mx\epsilon^{\mp mz}(\epsilon^{mc} + \epsilon^{-mc}) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89);$$

the upper or lower sign of the index being taken according as z is positive or negative.

These potentials must be added to those formerly obtained, to get the potential at any point after displacement.

We have next to calculate the pressure of the fluid at any point, on the supposition that the imaginary planes protect each slice of the fluid from the pressure of the adjacent slices, so that it is in equilibrium under the action of the forces of attraction, and the pressure of these planes on each side. Now in a fluid of density k, in equilibrium under forces whose potential is V, we have always—

$$\frac{dp}{dV} = k;$$

so that if we know that the value of p is p_0 where that of V is V_0, then at any other point

$$p = p_0 + k(V - V_0).$$

Now, at the free surface of the fluid, $p = 0$, and the distance from the free surface of the disturbed fluid to the plane of the original surface is ζ, a small quantity. The attraction which acts on this stratum of fluid is, in the first place, that of the undisturbed stratum, and this is equal to $4\pi kc$, towards that stratum. The pressure due to this cause at the level of the original surface will be $4\pi k^2c\zeta$, and the pressure arising from the attractive forces due to the displacements upon this thin layer of fluid, will be small quantities of the second order, which we neglect. We thus find the pressure when $z = c$ to be,

$$p_0 = 4\pi k^2c^2mA \sin mx.$$

The potential of the undisturbed mass when $z = c$ is

$$V_0 = -2\pi kc^2,$$

and the potential of the disturbance itself for the same value of z, is

$$V_0' = 2\pi kcA \sin mx\,(1 + \epsilon^{-2mc}).$$

So that we find the general value of p at any other point to be

$$p = 2\pi k^2(c^2 - z^2) + 2\pi k^2cA \sin mx\left\{2cm - 1 - \epsilon^{-2mc} + \epsilon^{mc}(\epsilon^{mz} + \epsilon^{-mz})\right\} \quad (90).$$

This expression gives the pressure of the fluid at any point, as depending on the state of constraint produced by the displacement of the imaginary planes. The accelerating effect of these pressures on any particle, if it were allowed to move parallel to x, instead of being confined by the planes, would be,

$$-\frac{1}{k}\frac{dp}{dx}.$$

The accelerating effect of the attractions in the same direction is

$$\frac{dV}{dx},$$

so that the whole acceleration parallel to x is

$$X = -2\pi kmcA \cos mx\,(2mc - \epsilon^{-2mc} - 1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (91).$$

It is to be observed, that this quantity is independent of z, so that every particle in the slice, by the combined effect of pressure and attraction, is urged with the same force, and, if the imaginary planes were removed, each slice would move parallel to itself without distortion, as long as the absolute displacements remained small. We have now to consider the direction of the resultant force X, and its changes of magnitude.

We must remember that the original displacement is A cos mx, if therefore $(2mc - \epsilon^{-2mc} - 1)$ be positive, X will be opposed to the displacement, and the equilibrium will be stable, whereas if that quantity be negative, X will act along with the displacement and increase it, and so constitute an unstable condition.

It may be seen that large values of mc give positive results and small ones negative. The sign changes when

$$2mc = 1.147 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92),$$

which corresponds to a wave-length

$$\lambda = 2c\frac{2\pi}{1.147} = 2c\,(5.471) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93).$$

The length of the complete wave in the critical case is 5·471 times the thickness of the stratum. Waves shorter than this are stable, longer waves are unstable.

The quantity $\quad 2mc\,(2mc - \epsilon^{-2mc} - 1)$,

has a minimum when $\quad 2mc = \cdot607 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$,

and the wave length is 10·353 times the thickness of the stratum.

In this case $\quad 2mc\,(2mc - \epsilon^{-2mc} - 1) = -\cdot509 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$,

and $\quad X = \cdot509\pi kA \cos mx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$.

24. Let us now conceive that the stratum of fluid, instead of being infinite in extent, is limited in breadth to about 100 times its thickness. The pressures and attractions will not be much altered by this removal of a distant part of the stratum. Let us also suppose that this thin but broad strip is bent round in its own plane into a circular ring whose radius is more than ten times the breadth of the strip, and that the waves, instead of being exactly parallel to each other, have their ridges in the direction of radii of the ring. We shall then have transformed our stratum into one of Saturn's Rings, if we suppose those rings to be liquid, and that a considerable breadth of the ring has the same angular velocity.

Let us now investigate the conditions of stability by putting

$$x = -2\pi kmc\,(2mc - \epsilon^{-2mc} - 1)$$

into the equation for n. We know that x must lie between 0 and $\frac{\omega^2}{13{\cdot}9}$ to ensure stability. Now the greatest value of x in the fluid stratum is $\cdot 509\pi k$. Taking Laplace's ratio of the diameter of the ring to that of the planet, this gives 42·5 as the minimum value of the density of the planet divided by that of the fluid of the ring.

Now Laplace has shown that any value of this ratio greater than 1·3 is inconsistent with the rotation of any considerable breadth of the fluid at the same angular velocity, so that our hypothesis of a broad ring with uniform velocity is untenable.

But the stability of such a ring is impossible for another reason, namely, that for waves in which $2mc > 1.147$, x is negative, and the ring will be destroyed by these short waves in the manner described at page (36).

When the fluid ring is treated, not as a broad strip, but as a filament of circular or elliptic section, the mathematical difficulties are very much increased, but it may be shown that in this case also there will be a maximum value of x, which will require the density of the planet to be several times that of the ring, and that in all cases short waves will give rise to negative values of x, inconsistent with the stability of the ring.

It appears, therefore, that a ring composed of a continuous liquid mass cannot revolve about a central body without being broken up, but that the parts of such a broken ring may, under certain conditions, form a permanent ring of satellites.

On the Mutual Perturbations of Two Rings.

25. We shall assume that the difference of the mean radii of the rings is small compared with the radii themselves, but large compared with the distance of consecutive satellites of the same ring. We shall also assume that each ring separately satisfies the conditions of stability.

We have seen that the effect of a disturbing force on a ring is to produce a series of waves whose number and period correspond with those of the disturbing force which produces them, so that we have only to calculate the coefficient belonging to the wave from that of the disturbing force.

Hence in investigating the simultaneous motions of two rings, we may assume that the mutually disturbing waves travel with the same *absolute* angular velocity, and that a maximum in one corresponds either to a maximum or a minimum of the other, according as the coefficients have the same or opposite signs.

Since the motions of the particles of each ring are affected by the disturbance of the other ring, as well as of that to which they belong, the equations of motion of the two rings will be involved in each other, and the final equation for determining the wave-velocity will have eight roots instead of four. But as each of the rings has four *free* waves, we may suppose these to originate *forced* waves in the other ring, so that we may consider the eight waves of each ring as consisting of four free waves and four forced ones.

In strictness, however, the wave-velocity of the "free" waves will be affected by the existence of the forced waves which they produce in the other ring, so that none of the waves are really "free" in either ring independently, though the whole motion of the system of two rings as a whole is free.

We shall find, however, that it is best to consider the waves first as free, and then to determine the reaction of the other ring upon them, which is such as to alter the wave-velocity of both, as we shall see.

The forces due to the second ring may be separated into three parts.

1st. The constant attraction when both rings are at rest.

2nd. The variation of the attraction on the first ring, due to its own disturbances.

3rd. The variation of the attraction due to the disturbances of the second ring.

The first of these affects only the angular velocity. The second affects the waves of each ring independently, and the mutual action of the waves depends entirely on the third class of forces.

26. *To determine the attractions between two rings.*

Let R and a be the mass and radius of the exterior ring, R' and a' those of the interior, and let all quantities belonging to the interior ring be marked with accented letters. (Fig 5.)

1st. *Attraction between the Rings when at rest.*

Since the rings are at a distance small compared with their radii, we may calculate the attraction on a particle of the first ring as if the second were an infinite straight line at distance $a' - a$ from the first.

The mass of unit of length of the second ring is $\frac{R'}{2\pi a'}$, and the accelerating effect of the attraction of such a filament on an element of the first ring is

$$\frac{R'}{\pi a' (a - a')} \text{ inwards. } \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97).$$

The attraction of the first ring on the second may be found by transposing accented and unaccented letters.

In consequence of these forces, the outer ring will revolve faster, and the inner ring slower than would otherwise be the case. These forces enter into the *constant terms* of the equations of motion, and may be included in the value of K.

2nd. *Variation due to disturbance of first ring.*

If we put $a(1+\rho)$ for a in the last expression, we get the attraction when the first ring is displaced. The part depending on ρ is

$$-\frac{R'a}{\pi a'(a-a')^2}\rho \text{ inwards.} \quad \quad (98).$$

This is the only variation of force arising from the displacement of the first ring. It affects the value of L in the equations of motion.

3rd. *Variation due to waves in the second ring.*

On account of the waves, the second ring varies in distance from the first, and also in mass of unit of length, and each of these alterations produces variations both in the radial and tangential force, so that there are four things to be calculated :

1st. Radial force due to radial displacement.
2nd. Radial force due to tangential displacement.
3rd. Tangential force due to radial displacement.
4th. Tangential force due to tangential displacement.

1st. Put $a'(1+\rho')$ for a', and we get the term in ρ'

$$\frac{R'}{\pi a'}\,\frac{(2a'-a)}{(a'-a)^2}\rho' \text{ inwards} = \lambda'\rho', \text{ say.} \quad \quad (99).$$

2nd. By the tangential displacement of the second ring the section is reduced in the proportion of 1 to $1-\frac{d\sigma'}{ds'}$, and therefore there is an alteration of the radial force equal to

$$-\frac{R'}{\pi a'(a-a')}\,\frac{d\sigma'}{ds'} \text{ inwards} = -\mu'\frac{d\sigma'}{ds'}, \text{ say,} \quad \quad (100).$$

3rd. By the radial displacement of the second ring the direction of the filament near the part in question is altered, so that the attraction is no longer radial but forwards, and the tangential part of the force is

$$\frac{R'}{\pi a'(a-a')}\,\frac{d\rho'}{ds'} = +\mu'\frac{d\rho'}{ds'} \text{ forwards.} \quad \quad (101).$$

4th. By the tangential displacement of the second ring a tangential force arises, depending on the relation between the length of the waves and the distance between the rings.

If we make $m\dfrac{a-a'}{a'}=p$, and $m\displaystyle\int_{-\infty}^{+\infty}\frac{x\sin px\,dx}{(1+x^2)^{\frac{3}{2}}}=\Pi$,

the tangential force is

$$\frac{R'}{\pi a'(a-a')^2}\Pi\sigma'=\nu'\sigma'. \quad \ldots\ldots\ldots \quad (102).$$

We may now write down the values of λ, μ and ν by transposing accented and unaccented letters.

$$\lambda=\frac{R}{\pi a}\frac{(2a-a')}{(a-a')^2} \quad \mu=\frac{R}{\pi a(a'-a)} \quad \nu=\frac{R}{\pi a'(a-a')^2}\Pi. \quad \ldots \quad (103).$$

Comparing these values with those of λ' μ' and ν', it will be seen that the following relations are approximately true when a is nearly equal to a':

$$\frac{\lambda'}{\lambda}=-\frac{\mu'}{\mu}=\frac{\nu'}{\nu}=\frac{R'a}{Ra'}. \quad \ldots\ldots\ldots \quad (104).$$

27. To form the equation of motion.

The original equations were

$$\omega^2+\omega^2\rho+2\omega\frac{d\sigma}{dt}-\frac{d^2\rho}{dt^2}=P=S+K-(2S-L)A\rho-MB\rho+\lambda'\rho'-\mu'\frac{d\sigma'}{ds'},$$

$$2\omega\frac{d\rho}{dt}+\frac{d^2\sigma}{dt^2}=Q=MA\sigma+NB\sigma+\mu'\frac{d\rho'}{ds'}+\nu'\sigma'.$$

Putting
$$\rho=A\cos(ms+nt),\quad \sigma=B\sin(ms+nt),$$
$$\rho'=A'\cos(ms+nt),\ \sigma'=B'\sin(ms+nt),$$

then
$$\omega^2=S+K$$

$$\left.\begin{aligned}(\omega^2+2S+n^2-L)A+(2\omega n+M)B-\lambda'A'+\mu'mB'&=0\\ (2\omega n+M)A+(n^2+N)B-\mu'mA'+\nu'B'&=0\end{aligned}\right\}\quad \ldots \quad (105).$$

The corresponding equations for the second ring may be found by transposing accented and unaccented letters. We should then have four equations to determine the ratios of AB, $A'B'$, and a resultant equation of the eighth degree to determine n. But we may make use of a more convenient method, since λ', μ', and ν' are small. Eliminating B we find

$$\left.\begin{aligned}An^4-A(\omega^2+2K+L-N)n^2-4A\omega Mn+AN(3\omega^2)\\ (-\lambda'A'+\mu'mB')n^2+(\mu'mA'-\nu'B')2\omega n\end{aligned}\right\}=0 \quad \ldots \quad (106).$$

Putting
$$B=\beta A,\ A'=xA,\ B'=\beta'A'=\beta'xA,$$

we have
$$\left.\begin{aligned}n^4-\{\omega^2(+2\mathrm{K})+L-N\}n^2-4\omega Mn+3\omega^2N\\ +(-\lambda'+\mu'm\beta')n^2x+(\mu'm-\nu'\beta')2\omega nx\end{aligned}\right\}=U=0 \quad . \quad (107),$$

$$\frac{dU}{dn}=4n^3-2\omega^2n+\&\text{c}. \quad \ldots\ldots\ldots \quad (108),$$

$$\frac{dU}{dx}=-\lambda'n^2+\mu'm\beta'n^2+2\mu'm\omega n-2\nu'\beta'\omega n \quad \ldots\ldots \quad (109),$$

whence
$$\frac{dn}{dx} = \frac{\lambda' n - \mu' m\beta' n - 2\mu' m\omega + 2\nu'\beta'\omega}{4n^2 - 2\omega^2} \quad \text{. . . . .} \quad (110).$$

28. If we were to solve the equation for n, leaving out the terms involving x, we should find the wave-velocities of the four free waves of the first ring, supposing the second ring to be prevented from being disturbed. But in reality the waves in the first ring produce a disturbance in the second, and these in turn react upon the first ring, so that the wave-velocity is somewhat different from that which it would be in the supposed case. Now if x be the ratio of the radial amplitude of displacement in the second ring to that in the first, and if $\overline{n}$ be a value of n supposing $x = 0$, then by Maclaurin's theorem,

$$n = \overline{n} + \frac{dn}{dx}x \quad \text{. . . . .} \quad (111).$$

The wave-velocity relative to the ring is $-\frac{n}{m}$, and the absolute angular velocity of the wave in space is

$$\varpi = \omega - \frac{n}{m} = \omega - \frac{\overline{n}}{m} - \frac{1}{m}\frac{dn}{dx}x \quad \text{. . . . .} \quad (112),$$

$$= p - qx \quad \text{. . . . .} \quad (113),$$

where $p = \omega - \frac{\overline{n}}{m}$, and $q = \frac{1}{m}\frac{dn}{dx}$.

Similarly in the second ring we should have

$$\varpi' = p' - q'\frac{1}{x} \quad \text{. . . . .} \quad (114);$$

and since the corresponding waves in the two rings must have the same absolute angular velocity,

$$\varpi = \varpi', \text{ or } p - qx = p' - q'\frac{1}{x} \quad \text{. . . . .} \quad (115).$$

This is a quadratic equation in x, the roots of which are real when

$$(p - p')^2 + 4qq'$$

is positive. When this condition is not fulfilled, the roots are impossible, and the general solution of the equations of motion will contain exponential factors, indicating destructive oscillations in the rings.

Since q and q' are small quantities, the solution is always real whenever p and p' are considerably different. The absolute angular velocities of the two pairs of reacting waves, are then nearly

$$p + \frac{qq'}{p - p'}, \text{ and } p' - \frac{qq'}{p - p'},$$

instead of p and p', as they would have been if there had been no reaction of the forced wave upon the free wave which produces it.

When p and p' are equal or nearly equal, the character of the solution will depend on the sign of qq'. We must therefore determine the signs of q and q' in such cases.

Putting $\beta' = \frac{2\omega'}{n'}$, we may write the values of q and q'

$$\left.\begin{aligned} q &= \frac{n}{m} \cdot \frac{\lambda' + 2\mu' m\left(\frac{\omega'}{n'} - \frac{\omega}{n}\right) - 4\nu' \frac{\omega'}{n'}\frac{\omega}{n}}{4n^2 - 2\omega^2}, \\ q' &= \frac{n'}{m'} \cdot \frac{\lambda + 2\mu m\left(\frac{\omega}{n} - \frac{\omega'}{n'}\right) - 4\nu' \frac{\omega'}{n'}\frac{\omega}{n}}{4n'^2 - 2\omega'^2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (116).$$

Referring to the values of the disturbing forces, we find that

$$\frac{\lambda'}{\lambda} = -\frac{\mu'}{\mu} = \frac{\nu'}{\nu} = \frac{R'a}{Ra'}.$$

Hence
$$\frac{q}{q'} = \frac{n}{n'} \frac{4n'^2 - 2\omega'^2}{4n^2 - 2\omega^2} \cdot \frac{R'a}{Ra'}. \quad \ldots\ldots\ldots\ldots (117).$$

Since qq' is of the same sign as $\frac{q}{q'}$, we have only to determine whether $2n - \frac{\omega^2}{n}$, and $2n' - \frac{\omega'^2}{n}$, are of the same or of different signs. If these quantities are of the same sign, qq' is positive, if of different signs, qq' is negative.

Now there are four values of n, which give four corresponding values of $2n - \frac{\omega^2}{n}$:

$n_1 = -\omega + \&c.$, $\qquad 2n_1 - \frac{\omega^2}{n_1}$ is negative,

$n_2 = -$ a small quantity, $2n_2 - \frac{\omega^2}{n_2}$ is positive,

$n_3 = +$ a small quantity, $2n_3 - \frac{\omega^2}{n_3}$ is negative,

$n_4 = \omega - \&c.$, $\qquad 2n_4 - \frac{\omega^2}{n_4}$ is positive.

The quantity with which we have to do is therefore positive for the even orders of waves and negative for the odd ones, and the corresponding quantity in the other ring obeys the same law. Hence when the waves which act upon each other are either both of even or both of odd names, qq' will be positive, but when one belongs to an even series, and the other to an odd series, qq' is negative.

29. The values of p and p' are, roughly,

$$\left.\begin{aligned} p_1 &= \omega + \frac{\omega}{m} - \&c., \quad p_2 = \omega + \&c., \quad p_3 = \omega - \&c., \quad p_4 = \omega - \frac{\omega}{m} + \&c., \\ p_1' &= \omega' + \frac{\omega'}{m} - \&c., \quad p_2' = \omega' + \&c., \quad p_3' = \omega' - \&c., \quad p_4' = \omega' - \frac{\omega'}{m} + \&c., \end{aligned}\right\} \quad \ldots (118).$$

ω' is greater than ω, so that p_1' is the greatest, and p_4 the least of these values, and of those of the same order, the accented is greater than the unaccented. The following cases of equality are therefore possible under suitable circumstances:

$$\begin{array}{ll} p_1 = p_3', & p_1 = p_2', \\ p_2 = p_4', & p_1 = p_4', \\ p_4 = p_4' \text{ (when } m = 1), & p_2 = p_3', \\ & p_3 = p_4'. \end{array}$$

In the cases in the first column qq' will be positive, in those in the second column qq' will be negative.

30. Now each of the four values of p is a function of m, the number of undulations in the ring, and of a the radius of the ring, varying nearly as $a^{-\frac{3}{2}}$. Hence m being given, we may alter the radius of the ring till any one of the four values of p becomes equal to a given quantity, say a given value of p', so that if an indefinite number of rings coexisted, so as to form a sheet of rings, it would be always possible to discover instances of the equality of p and p' among them. If such a case of equality belongs to the first column given above, two constant waves will arise in both rings, one travelling a little faster, and the other a little slower than the free waves. If the case belongs to the second column, two waves will also arise in each ring, but the one pair will gradually die away, and the other pair will increase in amplitude indefinitely, the one wave strengthening the other till at last both rings are thrown into confusion.

The only way in which such an occurrence can be avoided is by placing the rings at such a distance that no value of m shall give coincident values of p and p'. For instance, if $\omega' > 2\omega$, but $\omega' < 3\omega$, no such coincidence is possible. For p_1 is always less than p_2', it is greater than p_4 when $m = 1$ or 2, and less than p_4 when m is 3 or a greater number. There are of course an infinite number of ways in which this noncoincidence might be secured, but it is plain that if a number of concentric rings were placed at small intervals from each other, such coincidences must occur accurately or approximately between some pairs of rings, and if the value of $(p - p)^2$ is brought lower than $- 4qq'$, there will be destructive interference.

This investigation is applicable to any number of concentric rings, for, by the principle of superposition of small displacements, the reciprocal actions of any pair of rings are independent of all the rest.

31. *On the effect of long-continued disturbances on a system of rings.*

The result of our previous investigations has been to point out several ways in which disturbances may accumulate till collisions of the different particles of the rings take place. After such a collision the particles will still continue to revolve about the planet, but there will be a loss of energy in the system during the collision which can never be restored. Such collisions however will not affect what is called the Angular Momentum of the system about the planet, which will therefore remain constant.

Let M be the mass of the system of rings, and δm that of one ring whose radius is r, and angular velocity $\omega = S^{\frac{1}{2}} r^{-\frac{3}{2}}$. The angular momentum of the ring is

$$\omega r^2 \delta m = S^{\frac{1}{2}} r^{\frac{1}{2}} \delta m,$$

half its *vis viva* is $$\tfrac{1}{2}\omega^2 r^2 \delta m = \tfrac{1}{2} S r^{-1} \delta m.$$

The potential energy due to Saturn's attraction on the ring is

$$-\frac{S}{r}\, dm = -Sr^{-1}\delta m.$$

The angular momentum of the whole system is invariable, and is

$$S^{\frac{1}{2}} \Sigma\, (r^{\frac{1}{2}} \delta m) = A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (119).$$

The whole energy of the system is the sum of half the vis viva and the potential energy, and is

$$-\tfrac{1}{2} S \Sigma\, (r^{-1}\delta m) = E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (120).$$

A is invariable, while E necessarily diminishes. We shall find that as E diminishes, the distribution of the rings must be altered, some of the outer rings moving outwards, while the inner rings move inwards, so as either to spread out the whole system more, both on the outer and on the inner edge of the system, or, without affecting the extreme rings, to diminish the density or number of the rings at the mean distance, and increase it at or near the inner and outer edges.

Let us put $x = r^{\frac{1}{2}}$, then $A = S^{\frac{1}{2}} \Sigma\, (x dm)$ is constant.

Now let $$x_1 = \frac{\Sigma\, (x dm)}{\Sigma\, (dm)},$$

and $$x = x_1 + x',$$

then we may write

$$-\frac{2E}{S} = \Sigma\, (r^{-1}\, \delta m) = \Sigma\, (x^{-2}\, dm),$$

$$= \Sigma dm\, \left(x_1^{-2} - 2\frac{x'}{x_1^3} + 3\frac{x'^2}{x_1^4} - \&\text{c.}\right),$$

$$= \frac{1}{x_1^2} \Sigma\, (dm) - \frac{2}{x_1^3} \Sigma\, (x' dm) + \frac{3}{x_1^4} \Sigma\, (x'^2 \delta m) - \&\text{c.} \quad . \quad (121).$$

Now $\Sigma\, (dm) = M$ a constant, $\Sigma\, (x' dm) = 0$, and $\Sigma\, (x'^2 \delta m)$ is a quantity which increases when the rings are spread out from the mean distance either way, x' being subject only to the restriction $\Sigma\, (x' dm) = 0$. But $\Sigma\, (x'^2 dm)$ may increase without the extreme values of x' being increased, provided some other values be increased.

32. In fact, if we consider the very innermost particle as moving in an ellipse, and at the further apse of its orbit encountering another particle belonging to a larger orbit, we know

that the second particle, when at the same distance from the planet, moves the faster. The result is, that the interior satellite will receive a forward impulse at its further apse, and will move in a larger and less eccentric orbit than before. In the same way one of the outermost particles may receive a backward impulse at its nearer apse, and so be made to move in a smaller and less eccentric orbit than before. When we come to deal with collisions among bodies of unknown number, size, and shape, we can no longer trace the mathematical laws of their motion with any distinctness. All we can now do is to collect the results of our investigations and to make the best use we can of them in forming an opinion as to the constitution of the actual rings of Saturn which are still in existence and apparently in steady motion, whatever catastrophes may be indicated by the various theories we have attempted.

33. *To find the Loss of Energy due to internal friction in a broad Fluid Ring, the parts of which revolve about the Planet, each with the velocity of a satellite at the same distance.*

Conceive a fluid, the particles of which move parallel to the axis of x with a velocity u, u being a function of z, then there will be a tangential pressure on a plane parallel to xy

$$= \mu \frac{du}{dz} \text{ on unit of area}$$

due to the relative sliding of the parts of the fluid over each other.

In the case of the ring we have

$$\omega = S^{\frac{1}{2}} r^{-\frac{3}{2}}.$$

The absolute velocity of any particle is ωr. That of a particle at distance $(r + \delta r)$ is

$$\omega r + \frac{d}{dr}(\omega r)\,\delta r.$$

If the angular velocity had been uniform, there would have been no sliding, and the velocity would have been

$$\omega r + \omega \delta r.$$

The sliding is therefore

$$r \frac{d\omega}{dr} \delta r,$$

and the friction on unit of area perpendicular to r is $\mu r \dfrac{d\omega}{dr}$.

The loss of Energy, per unit of area, is the product of the sliding by the friction,

or $$\mu r^2 \left.\overline{\frac{d\omega}{dr}}\right|^2 \delta r \text{ in unit of time.}$$

The loss of Energy in a part of the Ring whose radius is r, breadth δr, and thickness c,

is $$2\pi r^3 c\mu \left.\overline{\frac{d\omega}{dr}}\right|^2 \delta r.$$

In the case before us it is $$\frac{9}{2}\pi\mu S c r^{-2}\delta r.$$

If the thickness of the ring is uniform between $r = a$ and $r = b$, the whole loss of Energy

is $$\frac{9}{2}\pi\mu S c\left(\frac{1}{b} - \frac{1}{a}\right),$$

in unit of time.

Now half the *vis viva* of an elementary ring is

$$\pi\rho c r\delta r\, r^2\omega^2 = \pi\rho c S\delta r,$$

and this between the limits $r = a$ and $r = b$ gives

$$\pi\rho c S(a - b).$$

The potential due to the attraction of S is twice this quantity with the sign changed, so that

$$E = -\pi\rho c S(a - b).$$

and $$\frac{dE}{dt} = \frac{9}{2}\pi\mu S\left(\frac{1}{b} - \frac{1}{a}\right),$$

$$\frac{1}{E}\frac{dE}{dt} = -\frac{9}{2}\frac{\mu}{\rho}\frac{1}{ab}.$$

Now Professor Stokes finds $\sqrt{\frac{\mu}{\rho}} = 0{\cdot}0564$ for water,

and $= 0{\cdot}116$ for air,

taking the unit of space one English inch, and the unit of time one second. We may take $a = 88{,}209$ miles, and $b = 77{,}636$ for the ring A; and $a = 75{,}845$ and $b = 58{,}660$ for the ring b. We may also take one year as the unit of time. The quantity representing the ratio of the loss of energy in a year to the whole energy is

$$\frac{1}{E}\frac{dE}{dt} = \frac{1}{60{,}880{,}000{,}000{,}000} \text{ for the ring } A,$$

$$\text{and } \frac{1}{39{,}540{,}000{,}000{,}000} \text{ for the ring } B,$$

showing that the effect of internal friction in a ring of water moving with steady motion is inappreciably small. It cannot be from this cause therefore that any decay can take place in the motion of the ring, provided that no waves arise to disturb the motion.

Recapitulation of the Theory of the Motion of a Rigid Ring.

The position of the ring relative to Saturn at any given instant is defined by three variable quantities.

1st. The distance between the centre of gravity of Saturn and the centre of gravity of the ring. This distance we denote by r.

2nd. The angle which the line r makes with a fixed line in the plane of the motion of the ring. This angle is called θ.

3rd. The angle between the line r and a line fixed with respect to the ring so that it coincides with r when the ring is in its mean position. This is the angle ϕ.

The values of these three quantities determine the position of the ring so far as its motion in its own plane is concerned. They may be referred to as the *radius vector*, *longitude*, and *angle of libration* of the ring.

The forces which act between the ring and the planet depend entirely upon their relative position. The method adopted above consists in determining the *potential* (V) of the ring at the centre of the planet in terms of r and ϕ. Then the *work done* by any displacement of the system is measured by the change of VS during that displacement. The attraction between the centre of gravity of the Ring and that of the planet is $-S\dfrac{dV}{dr}$, and the moment of the couple tending to turn the ring about its centre of gravity is $S\dfrac{dV}{d\phi}$.

It is proved in Problem V. that if a be the radius of a circular ring, $r_0 = af$ the distance of its centre of gravity from the centre of the circle, and R the mass of the ring, then, at the centre of the ring, $\dfrac{dV}{dr} = -\dfrac{R}{a^2}f$, $\dfrac{dV}{d\phi} = 0$.

It also appears that $\dfrac{d^2V}{dr^2} = \frac{1}{2}\dfrac{R}{a^3}(1+g)$, which is positive when $g > -1$,

and that $\dfrac{d^2V}{d\phi^2} = \frac{1}{2}\dfrac{R}{a}f^2(3-g)$, which is positive when $g < 3$.

If $\dfrac{d^2V}{dr^2}$ is positive, then the attraction between the centres decreases as the distance increases, so that, if the two centres were kept at rest at a given distance by a constant force, the equilibrium would be unstable. If $\dfrac{d^2V}{d\phi^2}$ is positive, then the forces tend to increase the angle of libration, in whichever direction the libration takes place, so that if the ring were fixed by an axis through its centre of gravity, its equilibrium round that axis would be unstable.

In the case of the uniform ring with a heavy particle on its circumference whose weight $= \cdot 82$ of the whole, the direction of the whole attractive force of the ring near the centre will pass through a point lying in the same radius as the centre of gravity, but at a distance from the centre $= \frac{9}{8} a$. (Fig. 6.)

If we call this point O, the line SO will indicate the direction and position of the force acting on the ring, which we may call F.

It is evident that the force F, acting on the ring in the line OS, will tend to turn it round its centre of gravity R and to increase the angle of libration KRO. The direct action of this force can never reduce the angle of libration to zero again. To understand the indirect action of the force, we must recollect that the centre of gravity (R) of the ring is revolving about Saturn in the direction of the arrows, and that the ring is revolving about its centre of gravity with nearly the same velocity. If the angular velocity of the centre of gravity about Saturn were always equal to the rotatory velocity of the ring, there would be no libration.

Now suppose that the angle of rotation of the ring is in advance of the longitude of its centre of gravity, so that the line RO has got in advance of SRK by the angle of libration KRO. The attraction between the planet and the ring is a force F acting in SO. We resolve this force into a couple, whose moment is $F{\cdot}RN$, and a force F acting through R the centre of gravity of the ring.

The couple affects the rotation of the ring, but not the position of its centre of gravity, and the force RF acts on the centre of gravity without affecting the rotation.

Now the couple, in the case represented in the figure, acts in the positive direction, so as to *increase* the angular velocity of the ring, which was already greater than the velocity of revolution of R about S, so that the angle of libration would increase, and never be reduced to zero.

The force RF does not act in the direction of S, but behind it, so that it becomes a retarding force acting upon the centre of gravity of the ring. Now the effect of a retarding force is to cause the distance of the revolving body to decrease and the angular velocity to increase, so that a retarding force increases the angular velocity of R about S.

The effect of the attraction along SO in the case of the figure is, first, to increase the rate of rotation of the ring round R, and secondly, to increase the angular velocity of R about S. If the second effect is greater than the first, then, although the line RO increases its angular velocity, SR will increase its angular velocity more, and will overtake RO, and restore the ring to its original position, so that SRO will be made a straight line as at first. If this accelerating effect is not greater than the acceleration of rotation about R due to the couple, then no compensation will take place, and the motion will be essentially unstable.

If in the figure we had drawn ϕ negative instead of positive, then the couple would have been negative, the tangential force on R accelerative, r would have increased, and in the cases of stability the retardation of θ would be greater than that of $(\theta + \phi)$, and the normal position would be restored, as before.

The object of the investigation is to find the conditions under which this compensation is possible.

It is evident that when SRO becomes straight, there is still a difference of angular velocities between the rotation of the ring and the revolution of the centre of gravity, so that there will be an oscillation on the other side, and the motion will proceed by alternate oscillations without limit.

If we begin with r at its mean value, and ϕ negative, then the rotation of the ring will be retarded, r will be increased, the revolution of r will be more retarded, and thus ϕ will be reduced to zero. The next part of the motion will reduce r to its mean value, and bring ϕ to its greatest positive value. Then r will diminish to its least value, and ϕ will vanish. Lastly r will return to the mean value, and ϕ to the greatest negative value.

It appears from the calculations, that there are, in general, two different ways in which this kind of motion may take place, and that these may have different periods, phases, and amplitudes. The mental exertion required in following out the results of a combined motion of this kind, with all the variations of force and velocity during a complete cycle, would be very great in proportion to the additional knowledge we should derive from the exercise.

The result of this theory of a rigid ring shows not only that a perfectly uniform ring cannot revolve permanently about the planet, but that the irregularity of a permanently revolving ring must be a very observable quantity, the distance between the centre of the ring and the centre of gravity being between ·8158 and ·8279 of the radius. As there is no appearance about the rings justifying a belief in so great an irregularity, the theory of the solidity of the rings becomes very improbable.

When we come to consider the additional difficulty of the tendency of the fluid or loose parts of the ring to accumulate at the thicker parts, and thus to destroy that nice adjustment of the load on which stability depends, we have another powerful argument against solidity.

And when we consider the immense size of the rings, and their comparative thinness, the absurdity of treating them as rigid bodies becomes self-evident. An iron ring of such a size would be not only plastic but semifluid under the forces which it would experience, and we have no reason to believe these rings to be artificially strengthened with any material unknown on this earth.

Recapitulation of the Theory of a Ring of equal Satellites.

In attempting to conceive of the disturbed motion of a ring of unconnected satellites, we have, in the first place, to devise a method of identifying each satellite at any given time, and in the second place, to express the motion of every satellite under the same general formula, in order that the mathematical methods may embrace the whole system of bodies at once.

By conceiving the ring of satellites arranged regularly in a circle, we may easily identify any satellite, by stating the angular distance between it and a known satellite when so arranged. If the motion of the ring were undisturbed, this angle would remain unchanged during the motion, but in reality, the satellite has its position altered in three ways: 1st, it may be further from or nearer to Saturn; 2ndly, it may be in advance or in the rear of the position it would have had if undisturbed; 3rdly, it may be on one side or other of the mean plane of the ring. Each of these displacements may vary in any way whatever as we pass from one satellite to another, so that it is impossible to assign beforehand the place of any satellite by knowing the places of the rest. § 2.

The formula, therefore, by which we are enabled to predict the place of every satellite at any given time, must be such as to allow the initial position of every satellite to be independent of the rest, and must express all future positions of that satellite by inserting the corresponding value of the quantity denoting time, and those of every other satellite by inserting the value of the angular distance of the given satellite from the point of reference. The three displacements of the satellite will therefore be functions of two variables—the angular position of the satellite, and the time. When the time alone is made to vary, we trace the complete motion of a single satellite; and when the time is made constant, and the angle is made to vary, we trace the form of the ring at a given time.

It is evident that the form of this function, in so far as it indicates the state of the whole ring at a given instant, must be wholly arbitrary, for the form of the ring and its motion at starting are limited only by the condition that the irregularities must be small. We have, however, the means of breaking up any function, however complicated, into a series of simple functions, so that the value of the function between certain limits may be accurately expressed as the sum of a series of sines and cosines of multiples of the variable. This method, due to Fourier, is peculiarly applicable to the case of a ring returning into itself, for the value of Fourier's series is necessarily periodic. We now regard the form of the disturbed ring at any instant as the result of the superposition of a number of separate disturbances, each of which is of the nature of a series of equal waves regularly arranged round the ring. Each of these elementary disturbances is characterised by the number of undulations in it, by their amplitude, and by the position of the first maximum in the ring. § 3.

When we know the form of each elementary disturbance, we may calculate the attraction of the disturbed ring on any given particle in terms of the constants belonging to that disturbance, so that as the actual displacement is the resultant of the elementary displacements, the actual attraction will be the resultant of the corresponding elementary attractions, and therefore the actual motion will be the resultant of all the motions arising from the elementary disturbances. We have therefore only to investigate the elementary disturbances one by one, and having established the theory of these, we calculate the actual motion by combining the series of motions so obtained.

Assuming the motion of the satellites in one of the elementary disturbances to be that of oscillation about a mean position, and the whole motion to be that of a uniformly revolving series of undulations, we find our supposition to be correct, provided a certain biquadratic equation is satisfied by the quantity denoting the rate of oscillation. § 6.

When the four roots of this equation are all real, the motion of each satellite is compounded of four different oscillations of different amplitudes and periods, and the motion of the whole ring consists of four series of undulations, travelling round the ring with different velocities. When any of these roots are impossible, the motion is no longer oscillatory, but tends to the rapid destruction of the ring.

To determine whether the motion of the ring is permanent, we must assure ourselves that the four roots of this equation are real, whatever be the number of undulations in the ring; for if any one of the possible elementary disturbances should lead to destructive oscillations, that disturbance might sooner or later commence, and the ring would be destroyed.

Now the number of undulations in the ring may be any whole number from one up to half the number of satellites. The forces from which danger is to be apprehended are greatest when the number of undulations is greatest, and by taking that number equal to half the number of satellites, we find the condition of stability to be

$$S > .4352\,\mu^2 R,$$

where S is the mass of the central body, R that of the ring, and μ the number of satellites of which it is composed. § 8. If the number of satellites be too great, destructive oscillations will commence, and finally some of the satellites will come into collision with each other and unite, so that the number of independent satellites will be reduced to that which the central body can retain and keep in discipline. When this has taken place, the satellites will not only be kept at the proper distance from the primary, but will be prevented by its preponderating mass from interfering with each other.

We next considered more carefully the case in which the mass of the ring is very small, so that the forces arising from the attraction of the ring are small compared with that due to the central body. In this case the values of the roots of the biquadratic are all real, and easily estimated. § 9.

If we consider the motion of any satellite about its mean position, as referred to axes fixed in the plane of the ring, we shall find that it describes an ellipse in the direction opposite to that of the revolution of the ring, the periodic time being to that of the ring as ω to n, and the tangential amplitude of oscillation being to the radial as 2ω to n. § 10.

The absolute motion of each satellite in space is nearly elliptic for the large values of n, the axis of the ellipse always advancing slowly in the direction of rotation. The path of a satellite corresponding to one of the small values of n is nearly circular, but the radius slowly increases and diminishes during a period of many revolutions. § 11.

The form of the ring at any instant is that of a re-entering curve, having m alternations of distance from the centre, symmetrically arranged, and m points of condensation, or crowding of the satellites, which coincide with the points of greatest distance when n is positive, and with the points nearest the centre when n is negative. § 12.

This system of undulations travels with an angular velocity $-\frac{n}{m}$ relative to the ring, and $\omega - \frac{n}{m}$ in space, so that during each oscillation of a satellite a complete wave passes over it. § 14.

To exhibit the movements of the satellites, I have made an arrangement by which 36 little ivory balls are made to go through the motions belonging to the first or fourth series of waves. (Figs. 7, 8.)

The instrument stands on a pillar A, in the upper part of which turns the cranked axle CC.

On the parallel parts of this axle are placed two wheels, RR and TT, each of which has 36 holes at equal distances in a circle near its circumference. The two circles are connected by 36 small cranks of the form KK, the extremities of which turn in the corresponding holes of the two wheels. That axle of the crank K which passes through the hole in the wheel S is bored, so as to hold the end of the bent wire which carries the satellite S. This wire may be turned in the hole so as to place the bent part carrying the satellite at any angle with the crank. A pin P, which passes through the top of the pillar, serves to prevent the cranked axle from turning; and a pin Q, passing through the pillar horizontally, may be made to fix the wheel R, by inserting it in a hole in one of the spokes of that wheel. There is also a handle H, which is in one piece with the wheel T, and serves to turn the axle.

Now suppose the pin P taken out, so as to allow the cranked axle to turn, and the pin Q inserted in its hole, so as to prevent the wheel R from revolving; then if the crank C be turned by means of the handle H, the wheel T will have its centre carried round in a vertical circle, but will remain parallel to itself during the whole motion, so that every point in its plane will describe an equal circle, and all the cranks K will be made to revolve exactly as the large crank C does. Each satellite will therefore revolve in a small circular orbit, in the same time with the handle H, but the position of each satellite in that orbit may be arranged as we please, according as we turn the wire which supports it in the end of the crank.

In fig 8, which gives a front view of the instrument, the satellites are so placed that each is turned 60^0 further round in its socket than the one behind it. As there are 36 satellites, this process will bring us back to our starting-point after 6 revolutions of the direction of the arm of the satellite; and therefore as we have gone round the ring once in the same direction, the arm of the satellite will have overtaken the radius of the ring five times.

Hence there will be five places where the satellites are beyond their mean distance from the centre of the ring, and five where they are within it, so that we have here a series of five undulations round the circumference of the ring. In this case the satellites are crowded together when nearest to the centre, so that the case is that of the *first* series of waves, when $m = 5$.

Now suppose the cranked axle C to be turned, and all the small cranks K to turn with it, as before explained, every satellite will then be carried round on its own arm in the same direction; but, since the direction of the arms of different satellites is different, their phases of revolution will preserve the same difference, and the system of satellites will still be arranged in five undulations, only the undulations will be propagated round the ring in the direction opposite to that of the revolution of the satellites.

To understand the motion better, let us conceive the centres of the orbits of the satellites to be arranged in a straight line instead of a circle, as in fig 10. Each satellite is here represented in a different phase of its orbit, so that as we pass from one to another from left to right, we find the position of the satellite in its orbit altering in the direction opposite to that of the hands of a watch. The satellites all lie in a trochoidal curve, indicated by the line through them in the figure. Now conceive every satellite to move in its orbit through a certain angle in the direction of the arrows. The satellites will then lie in the dotted line, the form of which is the same as that of the former curve, only shifted in the direction of the large arrow. It appears, therefore, that as the satellites revolve, the undulation travels, so that any part of it reaches successively each satellite as it comes into the same phase of rotation. It therefore travels from those satellites which are most advanced in phase to those which are less so, and passes over a complete wave-length in the time of one revolution of a satellite.

Now if the satellites be arranged as in fig. 8, where each is more advanced in phase as we go round the ring in the direction of rotation, the wave will travel in the direction opposite to that of rotation, but if they are arranged as in fig. 12, where each satellite is less advanced in phase as we go round the ring, the wave will travel in the direction of rotation. Fig. 8 represents the *first* series of waves where $m = 5$, and fig. 12 represents the *fourth* series where $m = 7$. By arranging the satellites in their sockets before starting, we might make m equal to any whole number, from 1 to 18. If we chose any number above 18 the result would be the same as if we had taken a number as much below 18 and changed the arrangement from the first wave to the fourth.

In this way we can exhibit the motions of the satellites in the first and fourth waves. In reality they ought to move in ellipses, the major axes being twice the minor, whereas in the machine they move in circles: but the character of the motion is the same, though the form of the orbit is different.

We may now show these motions of the satellites among each other, combined with the motion of rotation of the whole ring. For this purpose we put in the pin P, so as to prevent the crank axle from turning, and take out the pin Q so as to allow the wheel R to turn. If we then turn the wheel T, all the small cranks will remain parallel to the fixed crank, and the wheel R will revolve at the same rate as T. The arm of each satellite will continue parallel to itself during the motion, so that the satellite will describe a circle whose centre is at a distance from the centre of R, equal to the arm of the satellite, and measured in the same direction. In our theory of real satellites, each moves in an ellipse, having the central body in its focus, but this motion in an eccentric circle is sufficiently near for illustration. The motion of the waves relative to the ring is the same as before. The waves of the first kind travel faster than the ring itself, and overtake the satellites, those of the fourth kind travel slower, and are overtaken by them.

In fig. 11 we have an exaggerated representation of a ring of twelve satellites affected by a wave of the fourth kind where $m = 2$. The satellites here lie in an ellipse at any given instant, and as each moves round in its circle about its mean position, the ellipse also moves round in the same direction with half their angular velocity. In the figure the dotted line represents the position of the ellipse when each satellite has moved forward into the position represented by a dot.

Fig. 13 represents a wave of the first kind where $m = 2$. The satellites at any instant lie in an epitrochoid, which, as the satellites revolve about their mean positions, revolves in the opposite direction with half their angular velocity, so that when the satellites come into the positions represented by the dots, the curve in which they lie turns round in the opposite direction and forms the dotted curve.

In fig. 9 we have the same case as in fig. 13, only that the absolute orbits of the satellites in space are given, instead of their orbits about their mean positions in the ring. Here each moves about the central body in an eccentric circle, which in strictness ought to be an ellipse not differing much from the circle.

As the satellites move in their orbits in the direction of the arrows, the curve which they form revolves in the same direction with a velocity $1\frac{1}{2}$ times that of the ring.

By considering these figures, and still more by watching the actual motion of the ivory balls in the model, we may form a distinct notion of the motions of the particles of a discontinuous ring, although the motions of the model are circular and not elliptic. The model, represented on a scale of one-third in figs. 7 and 8, was made in brass by Messrs. Smith and Ramage of Aberdeen.

We are now able to understand the mechanical principle, on account of which a massive central body is enabled to govern a numerous assemblage of satellites, and to space them out into a regular ring; while a smaller central body would allow disturbances to arise among the individual satellites, and collisions to take place.

When we calculated the attractions among the satellites composing the ring, we found that if any satellite be displaced tangentially, the resultant attraction will draw it away from its mean position, for the attraction of the satellites it approaches will increase, while that of those it recedes from will diminish, so that its equilibrium when in the mean position is unstable with respect to tangential displacements; and therefore, since every satellite of the ring is statically unstable between its neighbours, the slightest disturbance would tend to produce collisions among the satellites, and to break up the ring into groups of conglomerated satellites.

But if we consider the dynamics of the problem, we shall find that this effect need not necessarily take place, and that this very force which tends towards destruction may become the condition of the preservation of the ring. Suppose the whole ring to be revolving round a central body, and that one satellite gets in advance of its mean position. It will then be attracted forwards, its path will become less concave towards the attracting body, so that its distance from that body will increase. At this increased distance its angular velocity will be less, so that instead of overtaking those in front, it may by this means be made to fall back to its original position. Whether it does so or not must depend on the actual values of the attractive forces and on the angular velocity of the ring. When the angular velocity is great and the attractive forces small, the compensating process will go on vigorously, and the ring will be preserved. When the angular velocity is small and the attractive forces of the ring great, the dynamical effect will not compensate for the disturbing action of the forces, and the ring will be destroyed.

If the satellite, instead of being displaced forwards, had been originally behind its mean position in the ring, the forces would have pulled it backwards, its path would have become more concave towards the centre, its distance from the centre would diminish, its angular velocity would increase, and it would gain upon the rest of the ring till it got in front of its mean position. This effect is of course dependent on the very same conditions as in the former case, and the actual effect on a disturbed satellite would be to make it describe an orbit about its mean position in the ring, so that if in advance of its mean position, it first recedes from the centre, then falls behind its mean position in the ring, then approaches the centre within the mean distance, then advances beyond its mean position, and, lastly, recedes from the centre till it reaches its starting-point, after which the process is repeated indefinitely, the orbit being always described in the direction opposite to that of the revolution of the ring.

We now understand what would happen to a disturbed satellite, if all the others were preserved from disturbance. But, since all the satellites are equally free, the motion of one will produce changes in the forces acting on the rest, and this will set them in motion, and this motion will be propagated from one satellite to another round the ring. Now propagated disturbances constitute waves, and all waves, however complicated, may be reduced to combinations of simple and regular waves; and therefore all the disturbances of the ring may be considered as the resultant of many series of waves, of different lengths, and travelling with different velocities. The investigation of the relation between the length and velocity of these waves forms the essential part of the problem, after which we have only to split up the original disturbance into its simple elements, to calculate the effect of each of these separately, and then to combine the results. The solution thus obtained will be perfectly general, and quite independent of the particular form of the ring, whether regular or irregular at starting. § 14.

We next investigated the effect upon the ring of an external disturbing force. Having split up the disturbing force into components of the same type with the waves of the ring (an operation which is always possible), we found that each term of the disturbing force generates a "forced wave" travelling with its own angular velocity. The magnitude of the forced wave depends not only on that of the disturbing force, but on the angular velocity with which the disturbance travels round the ring, being greater in proportion as this velocity more nearly coincides with that of one of the "free waves" of the ring. We also found that the displacement of the satellites was sometimes in the direction of the disturbing force, and sometimes in the opposite direction, according to the relative position of the forced wave among the four natural ones, producing in the one case positive, and in the other negative forced waves. In treating the problem generally, we must determine the forced waves belonging to every term of the disturbing force, and combine these with such a system of free waves as shall reproduce the initial state of the ring. The subsequent motion of the ring is that which would result from the free waves and forced waves together. The most important class of forced waves are those which are produced by waves in neighbouring rings. § 15.

We concluded the theory of a ring of satellites by tracing the process by which the ring would be destroyed if the conditions of stability were not fulfilled. We found two cases of instability, depending on the nature of the tangential force due to tangential displacement. If this force be in the direction opposite to the displacement, that is, if the parts of the ring are *statically stable*, the ring will be destroyed, the irregularities becoming larger and larger without being propagated round the ring. When the tangential force is in the direction of the tangential displacement, if it is below a certain value, the disturbances will be propagated round the ring without becoming larger, and we have the case of stability treated of at large.

If the force exceed this value, the disturbances will still travel round the ring, but they will increase in amplitude continually till the ring falls into confusion. § 18.

We then proceeded to extend our method to the case of rings of different constitutions. The first case was that of a ring of satellites of unequal size. If the central body be of sufficient mass, such a ring will be spaced out, so that the larger satellites will be at wider intervals than the smaller ones, and the waves of disturbance will be propagated as before, except that there may be reflected waves when a wave reaches a part of the ring where there is a change in the average size of the satellites. § 19.

The next case was that of an annular cloud of meteoric stones, revolving uniformly about the planet. The *average density* of the space through which these small bodies are scattered will vary with every irregularity of the motion, and this variation of density will produce variations in the forces acting upon the other parts of the cloud, and so disturbances will be propagated in this ring, as in a ring of a finite number of satellites. The condition that such a ring should be free from destructive oscillations is, that the density of the planet should be more than three hundred times that of the ring. This would make the ring much rarer than common air, as regards its *average density*, though the density of the particles of which it is composed may be great. Comparing this result with Laplace's minimum density of a ring revolving as a whole, we find that such a ring cannot revolve as a whole, but that the inner parts must have a greater angular velocity than the outer parts. § 20.

We next took up the case of a flattened ring, composed of incompressible fluid, and moving with uniform angular velocity. The internal forces here arise partly from attraction and partly from fluid pressure. We began by taking the case of an infinite stratum of fluid affected by regular waves, and found the accurate values of the forces in this case. For long waves the resultant force is in the same direction as the displacement, reaching a maximum for waves whose length is about ten times the thickness of the stratum. For waves about five times as long as the stratum is thick there is no resultant force, and for shorter waves the force is in the opposite direction to the displacement. § 23.

Applying these results to the case of the ring, we find that it will be destroyed by the long waves unless the fluid is less than $\frac{1}{42}$ of the density of the planet, and that in all cases the short waves will break up the ring into small satellites.

Passing to the case of *narrow* rings, we should find a somewhat larger maximum density, but we should still find that very short waves produce forces in the direction opposite to the displacement, and that therefore, as already explained (page 36), these short undulations would increase in magnitude without being propagated along the ring, till they had broken up the fluid filament into drops. These drops may or may not fulfil the condition formerly given for the stability of a ring of equal satellites. If they fulfil it, they will move as a permanent ring. If they do not, short waves will arise and be propagated among the satellites, with ever increasing magnitude, till a sufficient number of drops have been brought into collision, so as to unite and form a smaller number of larger drops, which may be capable of revolving as a permanent ring.

We have already investigated the disturbances produced by an external force independent of the ring; but the special case of the mutual perturbations of two concentric rings is considerably more complex, because the existence of a double system of waves changes the character of both, and the waves produced react on those that produced them.

We determined the attraction of a ring upon a particle of a concentric ring, first, when both rings are in their undisturbed state; secondly, when the particle is disturbed; and, thirdly, when the attracting ring is disturbed by a series of waves. § 26.

We then formed the equations of motion of one of the rings, taking in the disturbing forces arising from the existence of a wave in the other ring, and found the small variation of the velocity of a wave in the first ring as dependent on the magnitude of the wave in the second ring, which travels with it. § 27.

The forced wave in the second ring must have the same absolute angular velocity as the free wave of the first which produces it, but this velocity of the free wave is slightly altered by the reaction of the forced wave upon it. We find that if a free wave of the first ring has an absolute angular velocity not very different from that of a free wave of the second ring, then if both free waves be of even orders (that is, of the second or fourth varieties of waves), or both of odd orders (that is, of the first or third), then the swifter of the two free waves has its velocity increased by the forced wave which it produces, and the slower free wave is rendered still slower by its forced wave; and even when the two free waves have the same angular velocity, their mutual action will make them both split into two, one wave in each ring travelling faster, and the other wave in each ring travelling slower, than the rate with which they would move if they had not acted on each other.

But if one of the free waves be of an even order and the other of an odd order, the swifter free wave will travel slower, and the slower free wave will travel swifter, on account of the reaction of their respective forced waves. If the two free waves have naturally a certain small difference of velocities, they will be made to travel together, but if the difference is less than this, they will again split into two pairs of waves, one pair continually increasing in magnitude without limit, and the other continually diminishing, so that one of the waves in each ring will increase in violence till it has thrown the ring into a state of confusion.

There are four cases in which this may happen. The first wave of the outer ring may conspire with the second or the fourth of the inner ring, the second of the outer with the third of the inner, or the third of the outer with the fourth of the inner. That two rings may revolve permanently, their distances must be arranged so that none of these conspiracies may arise between odd and even waves, whatever be the value of m. The number of conditions to be fulfilled is therefore very great, especially when the rings are near together and have nearly the same angular velocity, because then there are a greater number of dangerous values of m to be provided for.

In the case of a large number of concentric rings, the stability of each pair must be investigated separately, and if in the case of any two, whether consecutive rings or not, there are a pair of conspiring waves, those two rings will be agitated more and more, till waves of that kind are rendered impossible by the breaking up of those rings into some different arrangement. The presence of the other rings cannot prevent the mutual destruction of any pair which bear such relations to each other.

It appears, therefore, that in a system of many concentric rings there will be continually new cases of mutual interference between different pairs of rings. The forces which excite these disturbances being very small, they will be slow of growth, and it is possible that by the irregularities of each of the rings the waves may be so broken and confused (see § 19), as to be incapable of mounting up to the height at which they would begin to destroy the arrangement of the ring. In this way it may be conceived to be possible that the gradual disarrangement of the system may be retarded or indefinitely postponed.

But supposing that these waves mount up so as to produce collisions among the particles, then we may deduce the result upon the system from general dynamical principles. There will be a tendency among the exterior rings to remove further from the planet, and among the interior rings to approach the planet, and this either by the extreme interior and exterior rings diverging from each other, or by intermediate parts of the system moving away from the mean ring. If the interior rings are observed to approach the planet, while it is known that none of the other rings have expanded, then the cause of the change cannot be the mutual action of the parts of the system, but the resistance of some medium in which the rings revolve. § 31.

There is another cause which would gradually act upon a broad fluid ring of which the parts revolve each with the angular velocity due to its distance from the planet, namely, the internal friction produced by the slipping of the concentric rings with different angular velocities. It appears, however (§ 33), that the effect of fluid friction would be insensible if the motion were regular.

Let us now gather together the conclusions we have been able to draw from the mathematical theory of various kinds of conceivable rings.

We found that the stability of the motion of a solid ring depended on so delicate an adjustment, and at the same time so unsymmetrical a distribution of mass, that even if the exact condition were fulfilled, it could scarcely last long, and if it did, the immense preponderance of one side of the ring would be easily observed, contrary to experience. These considerations, with others derived from the mechanical structure of so vast a body, compel us to abandon any theory of solid rings.

We next examined the motion of a ring of equal satellites, and found that if the mass of the planet is sufficient, any disturbances produced in the arrangement of the ring will be propagated round it in the form of waves, and will not introduce dangerous confusion. If the satellites are unequal, the propagation of the waves will no longer be regular, but disturbances of the ring will in this, as in the former case, produce only waves, and not growing confusion. Supposing the ring to consist, not of a single row of large satellites, but of a cloud of evenly distributed unconnected particles, we found that such a cloud must have a very small density in order to be permanent, and that this is inconsistent with its outer and inner parts moving with the same angular velocity. Supposing the ring to be fluid and continuous, we found that it will be necessarily broken up into small portions.

We conclude, therefore, that the rings must consist of disconnected particles; these may be either solid or liquid, but they must be independent. The entire system of rings must therefore consist either of a series of many concentric rings, each moving with its own velocity, and having its own systems of waves, or else of a confused multitude of revolving particles, not arranged in rings, and continually coming into collision with each other.

Taking the first case, we found that in an indefinite number of possible cases the mutual perturbations of two rings, stable in themselves, might mount up in time to a destructive magnitude, and that such cases must continually occur in an extensive system like that of Saturn, the only retarding cause being the possible irregularity of the rings.

The result of long-continued disturbance was found to be the spreading out of the rings in breadth, the outer rings pressing outwards, while the inner rings press inwards.

The final result, therefore, of the mechanical theory is, that the only system of rings which can exist is one composed of an indefinite number of unconnected particles, revolving round the planet with different velocities according to their respective distances. These particles may be arranged in series of narrow rings, or they may move through each other irregularly. In the first case the destruction of the system will be very slow, in the second case it will be more rapid, but there may be a tendency towards an arrangement in narrow rings, which may retard the process.

We are not able to ascertain by observation the constitution of the two outer divisions of the system of rings, but the inner ring is certainly transparent, for the limb of Saturn has been observed through it. It is also certain, that though the space occupied by the ring is transparent, it is not through the material parts of it that Saturn was seen, for his limb was observed without distortion; which shows that there was no refraction, and therefore that the rays did not pass through a medium at all, but between the solid or liquid particles of which the ring is composed. Here then we have an optical argument in favour of the theory of independent particles as the material of the rings. The two outer rings may be of the same nature, but not so exceedingly rare that a ray of light can pass through their whole thickness without encountering one of the particles.

Finally, the two outer rings have been observed for 200 years, and it appears, from the careful analysis of all the observations by M. Struvé, that the second ring is broader than when first observed, and that its inner edge is nearer the planet than formerly. The inner ring also is suspected to be approaching the planet ever since its discovery in 1850. These appearances seem to indicate the same slow progress of the rings towards separation which we found to be the result of theory, and the remark, that the inner edge of the inner ring is most distinct, seems to indicate that the approach towards the planet is less rapid near the edge, as we had reason to conjecture. As to the apparent unchangeableness of the exterior diameter of the outer ring, we must remember that the outer rings are certainly far more dense than the inner one, and that a small change in the outer rings must balance a great change in the inner one. It is possible, however, that some of the observed changes may be due to the existence of a resisting medium. If the changes already suspected should be confirmed by repeated observations with the same instruments, it will be worth while to investigate more carefully whether Saturn's Rings are permanent or transitionary elements of the Solar System, and whether in that part of the heavens we see celestial immutability, or terrestrial corruption and generation, and the old order giving place to new before our own eyes.

APPENDIX.

On the Stability of the Steady Motion of a Rigid Body about a Fixed Centre of Force. By PROFESSOR W. THOMSON, (*communicated in a letter*).

THE body will be supposed to be symmetrical on the two sides of a certain plane containing the centre of force, and no motion except that of parts of the body parallel to the plane will be considered. Taking it as the plane of construction, let G (fig. 14) be the centre of gravity of the body, and O a point at which the resultant attraction of the body is in the line OG towards G. Then if the body be placed with O coinciding with the centre of force, and set in a state of rotation about that point as an axis, with an angular velocity equal to $\sqrt{\dfrac{fS}{aM}}$, (where f denotes the attraction of the body on a unit of matter at O, S the amount of matter in the central body, M the mass of the revolving body, and a the distance OG), it will continue, provided it be perfectly undisturbed, to revolve uniformly at this rate, and the attraction Sf on the moving body will be constantly balanced by the centrifugal force $\omega^2 aM$ of its motion.

Let us now suppose the motion to be slightly disturbed, and let it be required to investigate the consequences. Let X, S, Y, be rectangular axes of reference revolving uniformly with the angular velocity ω, round S, the fixed attracting point. Let $\bar{x}$, $\bar{y}$, be the coordinates of G with reference to these axes, and let XS, YS denote the components of the whole force of attraction of S on the rigid body. Then since this force is in the line through S, its moment round G is

$$SY\bar{x} - SX\bar{y};$$

the components of the forces on the moving body being reckoned as *positive* when they tend to *diminish* $\bar{x}$ and $\bar{y}$ respectively. Hence if k denote the radius of gyration of the body round G, and if ϕ denote the angle which OG makes with SX (*i. e.* the angle GOK), the equations of motion are,

$$M\left(\frac{d^2\bar{x}}{dt^2} - 2\omega\frac{d\bar{y}}{dt} - \omega^2\bar{x}\right) + SX = 0,$$

$$M\left(\frac{d^2\bar{y}}{dt^2} + 2\omega\frac{d\bar{x}}{dt} - \omega^2\bar{y}\right) + SY = 0,$$

$$Mk^2\frac{d^2\phi}{dt^2} - S(Y\bar{x} - X\bar{y}) = 0.$$

In the first place we see that one integral of these equations is

$$M\left(\bar{x}\frac{d\bar{y}}{dt} - \bar{y}\frac{d\bar{x}}{dt}\right) + M\omega(\bar{x}^2 + \bar{y}^2) + Mk^2\frac{d\phi}{dt} = H.$$

This is the "equation of angular momentum."

In considering whether the motion round S with velocity ω when O coincides with S is stable or unstable, we must find whether every possible motion with the same "angular momentum" round S is such that it will never bring O to more than an infinitely small distance from S: that is to say, we must find whether, for every possible solution in which $H = M(a^2 + k^2)\,\omega$, and for which the co-ordinates of O are infinitely small at one time, these co-ordinates remain infinitely small. Let these values at time t be denoted thus: $SN = \xi$, and $NO = \eta$; let OG be at first infinitely nearly parallel to OX, *i.e.* let ϕ be infinitely small (the full solution will tell us whether or not ϕ remains infinitely small); then, as long as ϕ is infinitely small, we have

$$\bar{x} = a + \xi,\ \bar{y} = \eta + a\phi,$$

and the equations of motion have the forms

$$M\left\{\frac{d^2\xi}{dt^2} - 2\omega\left(\frac{d\eta}{dt} + a\frac{d\phi}{dt}\right) - \omega^2(a + \xi)\right\} + SX = 0,$$

$$M\left\{\frac{d^2\eta}{dt^2} + a\frac{d^2\phi}{dt^2} + 2\omega\frac{d\xi}{dt} - \omega^2(\eta + a\phi)\right\} + SY = 0,$$

and we may write the equation of angular momentum instead of the third equation,

$$M\left\{(a+\xi)\left(\frac{d\eta}{dt} + a\frac{d\phi}{dt}\right) - (\eta + a\phi)\left(\frac{d\xi}{dt}\right) + \omega(a+\xi)^2 + \omega(\eta + a\phi)^2 + k^2\frac{d\phi}{dt}\right\} = H.$$

If now we suppose ξ and η to be infinitely small, the last of these equations becomes

$$(a^2 + k^2)\frac{d\phi}{dt} + 2\omega a\xi + a\frac{d\eta}{dt} = 0\ \ldots\ldots\ldots\ldots \quad (a).$$

If p and q denote the components parallel and perpendicular to OG of the attraction of the body on a unit of matter at S, we have

$$X = p\cos\phi - q\sin\phi = p,\ \text{and}\ Y = p\sin\phi + q\cos\phi = p\phi + q,$$

since q and ϕ are each infinitely small; and if we put V = potential at S, and

$$\alpha = \frac{d^2V}{d\xi^2},\ \beta = \frac{d^2V}{d\eta^2},\ \gamma = \frac{d^2V}{d\xi d\eta},$$

then

$$p = f - \alpha\xi - \gamma\eta,\ q = -\beta\eta - \gamma\xi,$$

$$X = f - \alpha\xi - \gamma\eta,\ Y = f\phi - \beta\eta - \gamma\xi.$$

If we make these substitutions for X and Y, and take into account that

$$f = \omega^2 a\frac{M}{S}\ \ldots\ldots\ldots\ldots \quad (b),$$

the first and second equations of motion become

$$\frac{d^2\xi}{dt^2} - 2\omega\frac{d\eta}{dt} - \omega^2\xi - 2\omega a\frac{d\phi}{dt} - \frac{S}{M}(\alpha\xi + \gamma\eta) = 0\ \ldots\ldots \quad (c),$$

$$\frac{d^2\eta}{dt^2} + 2\omega\frac{d\xi}{dt} - \omega^2\eta + a\frac{d^2\phi}{dt^2} - \frac{S}{M}(\beta\eta + \gamma\xi) = 0\ \ldots\ldots \quad (d).$$

Combining equations (a), (c), and (d), by the same method as that adopted in the text, we find that the differential equation in ξ, η, or ϕ, is of the form

$$A\frac{d^4u}{dt^4} + B\frac{d^2u}{dt^2} + Cu = 0,$$

where $A = k^2$,

$$B = \omega^2(2k^2 - a^2) - \frac{S}{M}\{k^2\alpha + (a^2 + k^2)\beta\},$$

$$C = \omega^4(k^2 - 3a^2) + \omega^2\frac{S}{M}\{(a^2 + k^2)(\alpha + \beta) - 4a^2\beta\} + (a^2 + k^2)\frac{S^2}{M^2}(\alpha\beta - \gamma).$$

In comparing this result with that obtained in the Essay, we must put

r_0 for a,

R for M,

$R + S$ for S,

L for α,

Nr_0^2 for β,

Mr_0 for γ.

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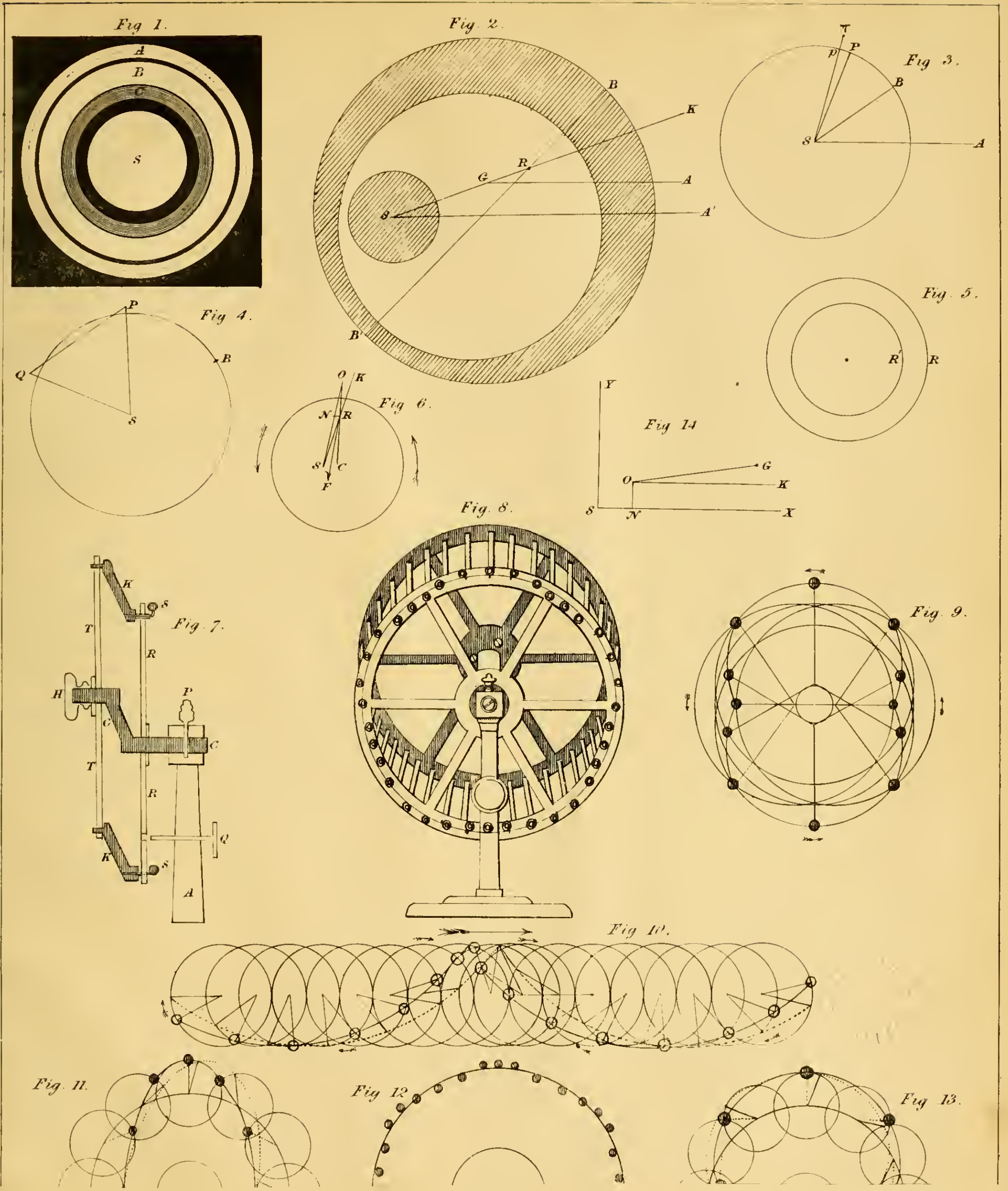
Fig 1.
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Fig. 2.
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Fig 3.
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Fig 4.
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Fig 6.
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Fig 14
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Fig. 7.
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Fig. 8.
Fig. 9.
Fig 10.
Fig. 11.
Fig 12.
Fig 13.